3.2 Functional relations

Meaning of Γ: Quantum analogue of the classical action

To see this, we derive the quantum equations of motion (Dyson-Schwinger eq.).

$$\boxed{\frac{1}{Z[J]}\int \mathcal{D}\varphi \, \frac{\delta}{\delta \varphi(x)}\left[e^{-S[\varphi] + \int_{x'} J(x')\varphi(x')}\right] = 0} \qquad (3.5)$$

(1) translat. inv. of $\mathcal{D}\varphi$
(2) no boundary terms

similarily to $\int_{\mathbb{R}} dx \frac{d}{dx}\left[e^{-S(x) + J\cdot x}\right] = 0$.

Performing the derivative in eq. (3.5), we get

$$\boxed{J(x) = \left\langle \frac{\delta S[\varphi]}{\delta \varphi(x)} \right\rangle_J} \qquad (3.8)$$

expectation value of classical EoM.

Eq. (3.6) is the Quantum EoM, the DSE.

In terms of Γ it reads (see eq. (3.4))

$$\boxed{\frac{\delta \Gamma}{\delta \phi(x)} = \left\langle \frac{\delta S[\varphi]}{\delta \varphi(x)} \right\rangle_{J_{max}(\phi)}} \qquad (3.9)$$

Computation of $\left\langle \frac{\delta S}{\delta \varphi} \right\rangle$ in ϕ^4-theory:

$$\frac{\delta S}{\delta \varphi(x)} = \left(-\Delta + m^2\right)\varphi(x) + \frac{\lambda}{3!}\left[\varphi(x)\right]^3 \qquad (3.10)$$

$\Phi(x) = \langle \varphi(x) \rangle$

$$\Rightarrow \left\langle \frac{\delta S}{\delta \varphi(x)} \right\rangle = \left(-\Delta + m^2\right)\Phi(x) + \frac{\lambda}{3!}\left\langle \varphi(x)^3 \right\rangle$$

$$= \frac{\delta S}{\delta \Phi(x)} + \frac{\lambda}{3!}\left(\left\langle \varphi(x)^3 \right\rangle - \Phi(x)^3 \right) \qquad (3.11)$$

Now we use: $\left\langle \varphi(x)^3 \right\rangle_J = \frac{1}{Z[J]} \int \mathcal{D}\varphi \; \varphi(x)^3 e^{-S[\varphi] + \int_x J \cdot \varphi}$

$$= \frac{1}{Z[J]} \frac{\delta}{\delta J(x)} \int \mathcal{D}\varphi \; \varphi(x)^2 e^{-S[\varphi] + \int_x J \cdot \varphi}$$

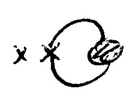
$\Phi(x) = \frac{1}{Z[J]} \int \mathcal{D}\varphi \, \varphi(x) e^{-S + \int J \varphi}$

$= \langle \varphi(x) \rangle_J$

$$= \left(\frac{\delta}{\delta J(x)} + \Phi(x) \right) \frac{1}{Z[J]} \int \mathcal{D}\varphi \; \varphi(x)^2 e^{-S[\varphi] + \int_x J \cdot \varphi}$$

$$= \left(\frac{\delta}{\delta J(x)} + \Phi(x) \right) \left[W^{(2)}[J](x,x) + \Phi(x)^2 \right] \qquad (3.12)$$

$\quad\; \shortparallel$



In general with iteration of eq. (3.12):

$$\boxed{\left\langle \prod_i \varphi(x_i) \right\rangle_J = \prod_i \left(\frac{\delta}{\delta J(x_i)} + \Phi(x_i) \right)} \qquad (3.13)$$

Applications:

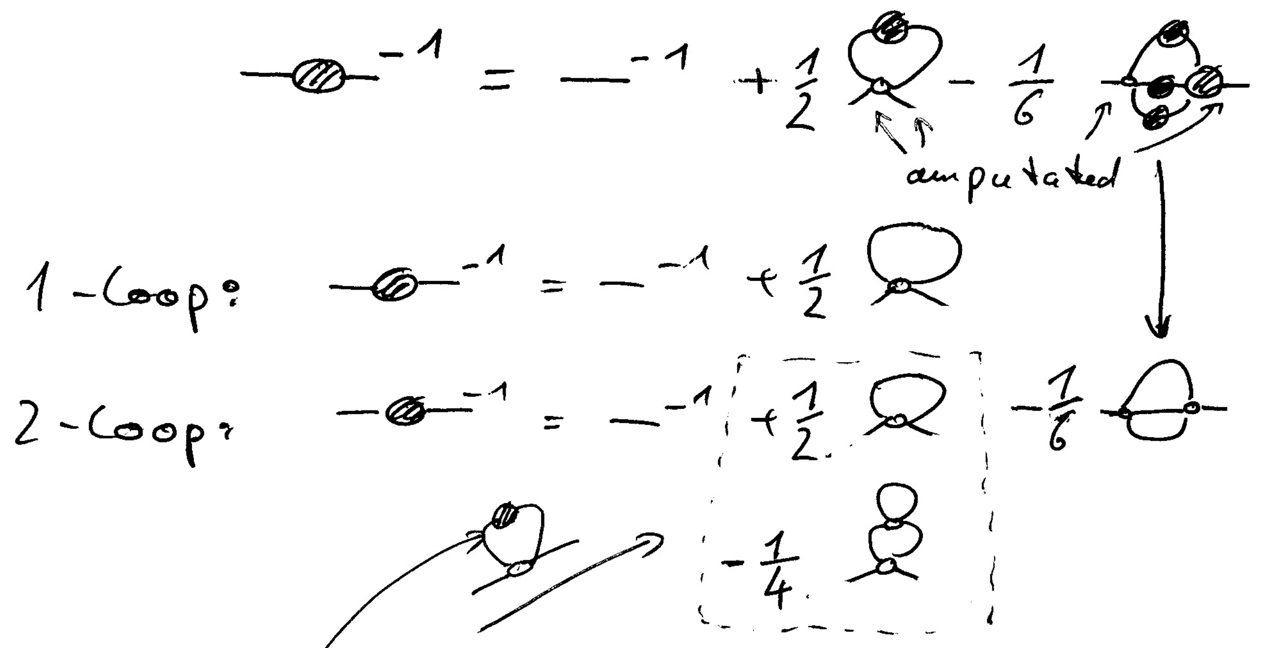
(a) Γ generates 1PI diagrams (in ϕ)

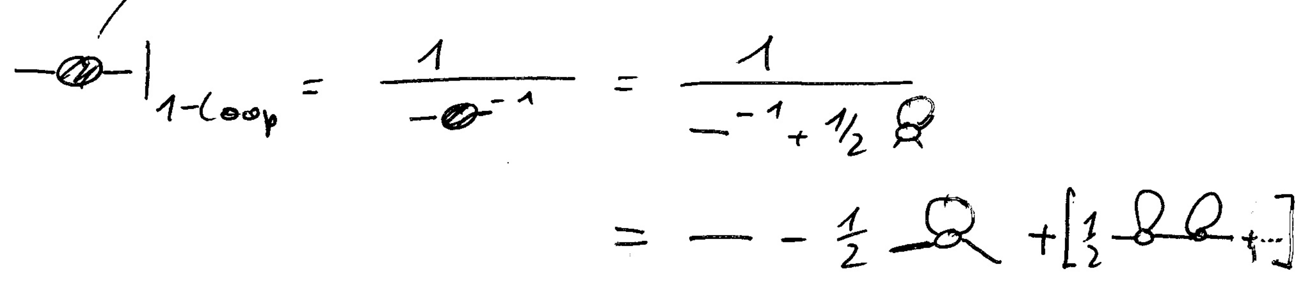
Induction: (i) classical action is 1PI

(ii) Assume n-loop Effective action is 1PI;

Insert into (3.16) $\square$

(b) Feynman diagrams: $\frac{\delta}{\delta\phi} \mathrm{DSE}\big|_{\phi=0}$



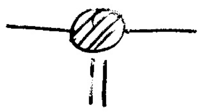


Analogously to eq. (3.6), (3.7) we have

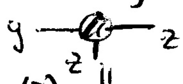
$$\frac{\delta}{\delta J(x)} = \int_y \overbrace{\frac{\delta \Phi(y)}{\delta J(x)}}^{W^{(2)}[\phi](y,x)} \frac{\delta}{\delta\phi(y)} \qquad (3.14)$$

$$\text{eq. (3.7)} \longrightarrow \quad = \int_y \frac{1}{\Gamma^{(2)}[\phi]}(x,y)\,\frac{\delta}{\delta\phi(y)}$$

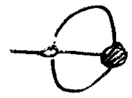
Inserting eq. (3.12) with (3.14) into eq. (3.11) we arrive at:



$$\left\langle \frac{\delta S}{\delta\varphi(x)} \right\rangle = \frac{\delta S}{\delta\Phi(x)} + \frac{\lambda}{3!}\left\{\left[\int_y \langle\varphi(x)\varphi(y)\rangle_c \frac{\delta}{\delta\Phi(y)} + \Phi(x)\right]\cdot\left(\frac{1}{\Gamma^{(2)}[\phi]}(x,x) + \Phi^2(x)\right) - \Phi^3(x)\right\}$$



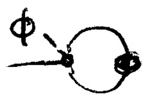
$$= \frac{\delta S}{\delta\phi(x)} - \frac{\lambda}{3!}\int_{y,z,z'} \langle\varphi(x)\varphi(y)\rangle_c\, \Gamma^{(3)}(y,z,z')$$



$$\frac{\delta}{\delta\phi(y)}\frac{1}{\Gamma^{(2)}}(x,x) = -\int_{z,z'} \frac{1}{\Gamma^{(2)}}(x,z)\,\Gamma^{(3)}(z,y,z')\,\frac{1}{\Gamma^{(2)}}(z',x) \longrightarrow$$

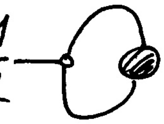
$$\cdot \langle\varphi(x)\varphi(z)\rangle_c \langle\varphi(x)\varphi(z')\rangle_c$$

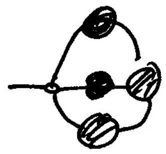
$$+ \frac{\lambda}{2}\langle\varphi(x)\varphi(x)\rangle_c\,\Phi(x) \qquad (3.15)$$



In summary:

$$\boxed{\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = \frac{\delta S[\phi]}{\delta\phi(x)} + \frac{1}{2}\ \text{(loop diagram)} - \frac{1}{3!}\ \text{(two-loop diagram)}} \qquad (3.16)$$





with


$n>2$ $\;= \Gamma^{(n)}[\phi](x_1, \ldots, x_n)$


n $\;= S^{(n)}[\phi](x_1, \ldots, x_n)$


$x \quad y$ $\;= \langle \varphi(x)\, \varphi(y) \rangle = \frac{1}{\Gamma^{(2)}[\phi]}(x, y)$

or

$$\boxed{\frac{\delta\Gamma}{\delta\phi(x)} = \frac{\delta S}{\delta\varphi(x)}\left[\varphi(x) = \int_y \frac{1}{\Gamma^{(2)}[\phi]}(x,y)\frac{\delta}{\delta\phi(y)} + \phi(y)\right]} \qquad (3.17)$$

Functional DSE

Eq (3.17) (or (3.16)) encodes the Quantum Equations of Motion (QEoM):

$$\boxed{\left.\frac{\delta\Gamma}{\delta\phi(x)}\right|_{\bar\phi} = 0} = \left\langle \frac{\delta S}{\delta\varphi} \right\rangle \qquad (3.18)$$