4.2 Generating functional

The generating functional, or here, the expectation value of observables are, in analogy to scalars and fermions, formally given by

$$\langle \hat{O}[A] \rangle = \frac{\int \mathcal{D}A \, \hat{O}[A] \, e^{-S_{YM}[A]}}{\int \mathcal{D}A \, e^{-S[A]}} \qquad (4.19)$$

pure YM

However, due to $\hat{O}[A^U] = \hat{O}[A]$ for observables, $\mathcal{D}A^U = \mathcal{D}A$ and $S_{YM}[A^U] = S_{YM}[A]$, the integral

U unitary

carries an infinite-dimensional redundancy.

Trivial example: $U(1)$ gauge theory

with $S_{U(1)}[A] = \frac{1}{4}\int d^dx \, F_{\mu\nu} F_{\mu\nu}$

$$= \frac{1}{2}\int d^dx \left(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu\right) \qquad (4.20)$$

free theory

We write $A_\nu = A_\nu^{gf\,\mathcal{U}}$ with A_ν^{gf} (4.21)

satisfies some gauge. To be specific, let's take Landau gauge:

$$\partial_\nu A_\nu^{gf} = 0 \tag{4.22}$$

$$\text{and} \quad A_\nu^{gf\,\mathcal{U}} = A_\nu^{gf} + \frac{1}{g}\partial_\nu \omega$$

We perform a change of variables:

$$A_\nu \to (A_\nu^{gf}, \omega) \tag{4.23}$$

and hence

$$\mathcal{D}A = \mathcal{D}A^{gf} \cdot \mathcal{D}\omega \cdot J \tag{4.24}$$

with Jacobi determinant J. We implement eq. (4.24) by inserting a 1: $\mathcal{U} = e^{i\omega}$

$$1 = \Delta_{\mathcal{F}}[A] \int \mathcal{D}\omega \; \delta[\mathcal{F}(A^{\mathcal{U}(\omega)})]$$

with (4.25)

$$\mathcal{F} = \partial_\nu A_\nu$$

Note that

$$\Delta_{\mathcal{F}}[A] = \left(\int \mathcal{D}\omega \; \delta[\mathcal{F}(A^{\mathcal{U}})]\right)^{-1}$$

Faddeev-Popov trick

$\Delta_{\mathcal{F}}[A]$ is gauge invariant by definition,

as $\displaystyle\int \mathcal{D}\omega\, \delta[\mathcal{F}[(A^V)^{U(\omega)}]]$, $V = e^{i\omega_V}$

$$= \int \mathcal{D}\omega\, \delta[\mathcal{F}[A^{V\cdot U(\omega)}]] = \int d\omega\, \delta[\mathcal{F}[A^{U(\omega')}]]$$

$\omega' = \omega + \omega_V$

$$\underset{\mathcal{D}\omega' = \mathcal{D}\omega}{=} \int \mathcal{D}\omega\, \delta[\mathcal{F}[A^{U(\omega)}]] \tag{4.26}$$

$\bar{A} = A^{U(\omega_0)}$, $\mathcal{F}[A^{U(\omega_0)}] = 0$

We use that

$$\delta[\mathcal{F}[A^{U(\omega)}]] = \frac{1}{\left|\det \frac{\delta \mathcal{F}}{\delta\omega}\right|(A^U)}\,\delta[\omega - \omega_0] \tag{4.27}$$

assuming unique solution ω_0

and hence

$$\boxed{\Delta_{\mathcal{F}}[A] = \left|\det \frac{\delta\mathcal{F}}{\delta\omega}\right|(A^{U(\omega_0)})} \tag{4.28}$$

Faddeev-Popov det.

In Landau gauge:

$$\mathcal{F}[A^{U(\omega)}] = \partial_\nu A_\nu + \frac{1}{g}\partial_\nu\partial_\nu\omega \tag{4.29}$$

It follows that

$$g \, \frac{\delta \mathcal{F}^{u}}{\delta \omega}(x,y) = \partial_\nu \partial_\nu \, \delta^d(x-y) \tag{4.30}$$

and the normalisation $\Delta_{\mathcal{F}}[A]$ follows as

$$\begin{aligned} \Delta_{\partial_\nu A_\nu}[A] &\simeq \left|\det\left(-\partial_\nu \partial_\nu\right)\right| \\ &= \det\left(-\partial_\nu\partial_\nu\right) \end{aligned} \tag{4.31}$$

$-\partial_\nu\partial_\nu > 0$

Finally we have for observables, eq. (4.19),

$$\langle \hat{O} \rangle = \frac{\int \mathcal{D}A \, \hat{O}[A] e^{-S_{U(1)}[A]}}{\int \mathcal{D}A \, e^{-S_{U(1)}[A]}} \tag{4.32}$$

$$= \frac{\int \mathcal{D}A \, \mathcal{D}\omega \; \Delta_{\mathcal{F}}[A] \, \delta[\mathcal{F}[A^u]] \, O[A] e^{-S_{U(1)}}}{\int \mathcal{D}A \, \mathcal{D}\omega \; \Delta_{\mathcal{F}}[A] \, \delta[\mathcal{F}[A^u]] \, e^{-S_{U(1)}}}$$

$$= \frac{\int \mathcal{D}\bar{A} \, \cancel{\mathcal{D}\omega} \; \Delta_{\mathcal{F}}[\bar{A}] \, \delta[\mathcal{F}[\bar{A}]] \, O[A] e^{-S_{U(1)}}}{\int \mathcal{D}\bar{A} \, \cancel{\mathcal{D}\omega} \; \Delta_{\mathcal{F}}[\bar{A}] \, \delta[\mathcal{F}[\bar{A}]] \, e^{-S_{U(1)}}}$$

with $A = \bar{A}^{u^{-1}}$ and $\mathcal{D}\bar{A}^{u^{-1}} = \mathcal{D}\bar{A}$, $\Delta_{\mathcal{F}}[\bar{A}^{u}] = \Delta_{\mathcal{F}}[A]$,

$S_{U(1)}[\bar{A}^{u^{-1}}] = S_{U(1)}[\bar{A}]$ and $\mathcal{F}[\bar{A}^{u^{-1}u}] = \mathcal{F}[\bar{A}]$.

In summary

$$\boxed{\langle \hat{O} \rangle = \frac{\int \mathcal{D}A\, \delta[\mathcal{F}(A)]\, \Delta_{\mathcal{F}}[A]\, \hat{O}[A]\, e^{-S_{U(1)}[A]}}{\int \mathcal{D}A\, \delta[\mathcal{F}(A)]\, \Delta_{\mathcal{F}}[A]\, e^{-S_{U(1)}[A]}}} \tag{4.33}$$

Average gauges: instead of eq. (4.25) use

$$\text{const.} = \int \mathcal{D}\omega \mathcal{D}\mathcal{C} \left| \det \underbrace{\frac{\delta \mathcal{F}}{\delta \omega}}_{\frac{\delta \mathcal{F}}{\delta \omega}\big|_{\omega=0}(A^u)} \right| (A^u)\, \delta[\mathcal{F}(A^u) - \mathcal{C}]\, e^{-\frac{1}{2\xi}\int d^d x\, \mathcal{C}(x)^2} \tag{4.34}$$

Everything follows accordingly, only $\omega_0 = \omega_0(\mathcal{C})$.

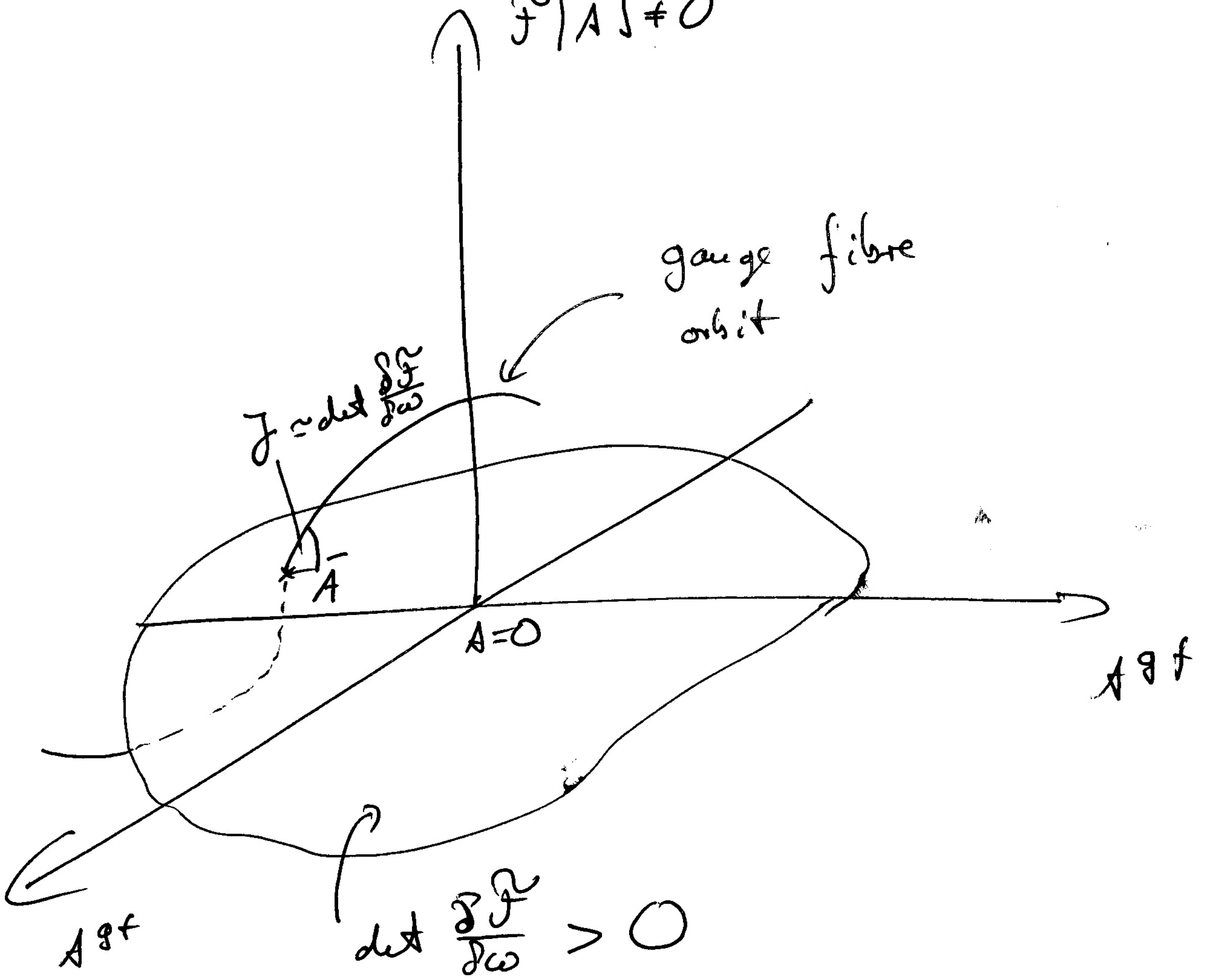
However, $\omega_0(\mathcal{C})$ drops out, and we arrive at

$$\langle \hat{O} \rangle = \frac{1}{N} \int \mathcal{D}A\, \mathcal{D}\mathcal{C} \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right| (A)\, \delta[\mathcal{F}(A) - \mathcal{C}]\, e^{-\frac{1}{2\xi}\int \mathcal{C}^2} \cdot \hat{O}[A]\, e^{-S_{U(1)}[A]}$$

$$= \frac{1}{N} \int \mathcal{D}A\, \hat{O}[A] \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right| (A)\, e^{-\left(S_{U(1)}[A] + S_{gf}[A]\right)}$$

with

$$S_{gf}[A] = \frac{1}{2\xi} \int d^d x\, \mathcal{F}(A)^2 \tag{4.35}$$


$\tilde{\mathcal{F}}[A] \neq 0$
gauge fibre
orbit
$J = \det \frac{\delta \tilde{\mathcal{F}}}{\delta \omega}$
$\bar{A}$
$A = 0$
A^{gf}
A^{gf}
$\det \frac{\delta \tilde{\mathcal{F}}}{\delta \omega} > 0$

Remarks:

(i) $\det \frac{\delta \mathcal{F}}{\delta \omega}$ does not depend on the gauge field for $U(1)$ in linear gauges, $\mathcal{F} = l_\mu A_\mu$ with $l_\mu = \partial_\mu, n_\mu, \ldots$ ← const. vector

and can be dropped.

(ii) The final action

$$S[A] = S_{U(1)}[A] + \frac{1}{2\xi} \int d^dx \left(\partial_\mu A_\mu\right)^2 \tag{4.36}$$

is that used in QFT I in the Gupta-Bleuler quantisation.

(iii) Generating functional:

$$Z[J] = \int \mathcal{D}A \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right| \cdot e^{-S_{U(1)}[A] - \frac{1}{2\xi}\int_x (\partial_\mu A_\mu)^2} \cdot e^{\int_x J_\mu A_\mu} \tag{4.37}$$

is gauge-dependent, but

$$\langle \hat{O} \rangle = \frac{1}{Z[0]} \left(\hat{O}\left[\frac{\delta}{\delta J}\right] Z[J] \right)_{J=0}$$ are not!

Non-Abelian case: the derivation in the $U(1)$ case holds true. None of the steps was specific to $U(1)$. However, the FP-det is gauge field-dep.:

up to det $1/g$ $\quad \mathcal{F}(\bar{A}) - \mathcal{C} = 0$

$$\Delta_{\mathcal{F}}[A] = \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right|_{\omega=\omega_0} \simeq \left| \det \mathcal{M} \right| [\bar{A}] \Big|$$

with

$$\mathcal{M}^{ab}(x,y) \simeq g \frac{\delta \mathcal{F}^a(A(x))}{\delta \omega^b(y)}\Bigg|_{\omega=0} = g \frac{\delta \mathcal{F}^a(A + \frac{1}{g} D\omega)}{\delta \omega^b(y)}$$

$$= \int_z \frac{\delta \mathcal{F}^a(A(x))}{\delta A^c_\nu(z)} \frac{\delta D^{cd}_{\nu,z} \omega^d(z)}{\delta \omega^b(y)}$$

$$= \int_z \frac{\delta \mathcal{F}^a(A(x))}{\delta A^c_\nu(z)} D^{cb}_{\nu,z} \delta^d(z-y) \tag{4.38}$$

In Lorentz gauge: $\mathcal{M}^{ab}(x,y) = -\partial_\nu D^{ab}_\nu \delta^d(x-y)$

We also have, see p. 76, eq. (4.34)

$$\Delta_{\mathcal{F}}[A]\, \delta[\mathcal{F}(A) - \mathcal{C}] = |\det \mathcal{M}|[\bar{A}]|\, \delta[\mathcal{F}(A) - \mathcal{C}]$$

$$= |\det \mathcal{M}|[A]|\, \delta[\mathcal{F}(A) - \mathcal{C}]$$

$$\text{with} \quad \mathcal{M} = -\frac{\delta \mathcal{F}}{\delta A_\nu} \cdot D_\nu \tag{4.39}$$

Finally

$$\boxed{\begin{gathered} Z[J] \simeq \int \mathcal{D}A \left|\det\left(-\frac{\delta \mathcal{F}}{\delta A_\nu}\cdot \mathcal{D}_\nu\right)\right| 1(A)\, e^{-S[A] + \int d^dx\, J^a_\nu \cdot A^a_\nu} \qquad (4.40) \\ \text{with} \quad S[A] = S_{YM}[A] + \underbrace{\frac{1}{2\xi}\int d^dx\, \mathcal{F}^a(A)\mathcal{F}^a(A)}_{S_{gf}[A]} \end{gathered}}$$

Example: $\mathcal{F}^a(A) = \partial_\nu A^a_\nu$ (4.41)

$$\Rightarrow S_{gf}[A] = \frac{1}{2\xi}\int_x \left(\partial_\nu A^a_\nu\right)^2$$

FP-det:

$$\det\left(-\frac{\delta \mathcal{F}}{\delta A_\nu}\mathcal{D}_\nu\right) = \det\left[-\partial_\nu \mathcal{D}_\nu\, \delta^d(x-y)\right]$$

$$= \det\left(-\partial_\nu \mathcal{D}_\nu\right) \qquad (4.42)$$

<u>Assume</u> now that $\mathcal{F}(A)$ has <u>one</u> solution, (does not hold for suff. smooth gauges (Gribov probl.)), and that $\mathcal{D}_\nu \frac{\delta \mathcal{F}}{\delta A_\nu}$ is positive (does not hold for ...)

Then we can use

$$\left| \det\left(-\frac{\delta \mathcal{F}}{\delta A_\nu}\cdot \mathcal{D}_\nu\right)\right| = \det\left(-\frac{\delta \mathcal{F}}{\delta A_\nu}\cdot \mathcal{D}_\nu\right)$$

$$\simeq \int \mathcal{D}c\,\mathcal{D}\bar{c}\; e^{-S_{gh}[c,\bar{c},A]} \qquad (4.43)$$

fermionic ghost

with

$$S_{gh}[c,\bar{c},A] = -\int d^dx\, d^dy\; \bar{c}^a(x)\, \frac{\delta \mathcal{F}^a}{\delta A_\nu^c}\cdot \mathcal{D}_\nu^{cb}\; c^b(y)$$

For Lorentz gauge:

$$S_{gh}[c,\bar{c},A] = -\int d^dx\; \bar{c}^a \partial_\nu \mathcal{D}_\nu^{ab} c^b \qquad (4.44)$$

Comments:

(i) the ghost has 'negative' propagator

if

$$\boxed{+}\int \bar{c}\, \partial_\nu \mathcal{D}_\nu c \qquad\qquad G_c(p) = \boxed{-}\, \frac{1}{p^2}\, \delta^{ab} \qquad (4.45)$$

(ii) the ghost does not obey the spin-statistics theorem

In summary,

$$Z[J,\eta,\bar\eta] = \int \mathcal{D}A\,\mathcal{D}c\,\mathcal{D}\bar c\; e^{-S[A,c,\bar c]}$$

$$\cdot\, e^{\int_x \left(J^a_\nu A^a_\nu + \bar\eta c - \bar c \eta\right)} \tag{4.46}$$

with $(\mathcal{F}^a(A) = \partial_\nu A^a_\nu)$

$$S[A,c,\bar c] = S_{YM}[A] + S_{gh}[c,\bar c, A] + S_{gf}[A]$$

$$S_{YM}[A] = \frac{1}{4}\int_x F^a_{\nu\nu} F^a_{\nu\nu}$$

$$S_{gh}[c,\bar c, A] = -\int_x \bar c^a\, \partial_\nu D^{ab}_\nu c^b \tag{4.47}$$

$$S_{gf}[A] = \frac{1}{2\xi}\int_x \left(\partial_\nu A^a_\nu\right)^2$$

and $F^a_{\nu\nu}$ given in eq. (4.11),

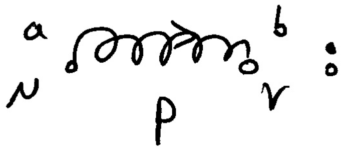
$$F^a_{\nu\nu} = \partial_\nu A^a_\nu - \partial_\nu A^a_\nu + g f^{abc} A^b_\nu A^c_\nu \tag{4.48}$$

For a general gauge $\mathcal{F}^a(A)$ we have

$$S_{gh} \simeq -\int \bar c\, \frac{\delta \mathcal{F}}{\delta A_\nu} D_\nu c\,, \quad S_{gf} = \frac{1}{2\xi}\int \mathcal{F}^{a\,2} \tag{4.49}$$

Feynman rules: (in Lorentz gauge)

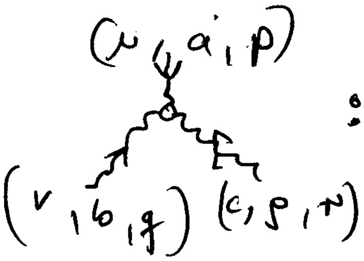
Propagators:

gluon: 

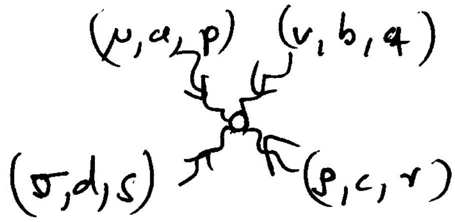
: $$\frac{1}{p^2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}(1-\xi)\right)\delta^{ab} \qquad (4.50a)$$

ghost: $\underset{a}{\circ} \overset{}{-\!\!\!\rightarrow\!\!\!-}_{p} \underset{b}{\circ}$: $$\frac{1}{p^2}\,\delta^{ab} \qquad (4.50b)$$

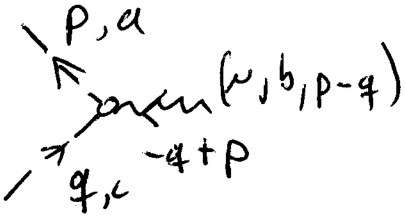
Vertices:



: $$i g f^{abc} \cdot \left(\delta_{\mu\nu}(p-q)_\rho + \delta_{\nu\rho}(q-r)_\mu + \delta_{\rho\mu}(r-p)_\nu\right) \qquad (4.51a)$$



: $$+ g^2 f^{abe} f^{cde} (2\pi)^d \delta^d(p+q+r+s) \cdot \left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \text{cycl. permuts in } \nu\rho\sigma\right) \qquad (4.51b)$$



: $$i g f^{abc}\, p_\mu \qquad (4.52)$$

Gauge invariance: the generating functional in eq. (4.46) is built upon the gauge-fixed action eq. (4.47). How does gauge-invariance manifest itself?

(i) Consider obervable $\langle \hat{O}[A] \rangle$ with

$\hat{O}[A^u] = \hat{O}[A]$: (4.53)

$$\frac{1}{N} \int \mathcal{D}A \; \hat{O}[A] \, e^{-S_{YM}[A]} = \frac{1}{N} \int \mathcal{D}A \; \hat{O}[A^u] \, e^{-S_{YM}[A]}$$

More generally: $f(A^u) \neq f(A)$ ($\langle f(A) \rangle$ no observ.)

$$\frac{1}{N} \int \mathcal{D}A \, f(A^u) \, e^{-S_{YM}[A]} \overset{A = \tilde{A}^{u^{-1}}}{=} \frac{1}{N} \int \mathcal{D}\tilde{A}^{u^{-1}} f(\tilde{A}^{u^{-1} u}) \, e^{-S_{YM}[\tilde{A}^{u^{-1}}]}$$

$$= \frac{1}{N} \int \mathcal{D}\tilde{A} \, f(\tilde{A}) \, e^{-S_{YM}[\tilde{A}]} \quad (4.54)$$

$$\Rightarrow \boxed{\frac{1}{N} \int \mathcal{D}A \left[f(A^u) - f(A) \right] e^{-S_{YM}[A]} = 0} \quad (4.55)$$

gauge inv. of S_{YM}
(and of $\mathcal{D}A$)