4.3 BRST-symmetry & unitarity

In the previous section we have introduced a functional integral approach, that reduces to Gupta-Bleuler in the case of $U(1)$. There we were able to define a positive-definite Hilbert space via projection onto the positive norm states, see QFT 1, chapter 6.2.

We have to extend such a projection procedure to the present $SU(N)$-case. Put differently, we have to extract the transversal gluons.

How does transversality ⟸ gauge symmetry manifest itself in the present case?

Classical gauge invariance:

$$S_{YM}[A^U] = S_{YM}[A] \qquad (4.56)$$

Infinitesimally: (see eq. (4.20), p. 71)

$$S_{YM}[A + \tfrac{1}{g} D\omega] + O(\omega^2) \qquad (4.57)$$

$$= S_{YM}[A] - \int_x \frac{1}{g}\,\omega^a D_\nu^{ab} \frac{\delta}{\delta A_\nu^b} S_{YM}[A] \;=\; S_{YM}[A]$$

It follows: ($\omega^a(x)$ general)

$$\boxed{D_\nu^{ab} \frac{\delta}{\delta A_\nu^b} S_{YM}[A] = 0,} \qquad (4.58)$$

that is, $D \cdot \frac{\delta}{\delta A}$ generates infinitesimal gauge transformations. In the U(1) case we have

$$\partial_\nu \frac{\delta}{\delta A_\nu} S_{U(1)}[A] = 0 \qquad (4.59)$$

Performing the derivative we get

$F_{\mu\nu} = \frac{1}{g i}[D_\mu, D_\nu]$

$$\boxed{\{D_\mu, D_\nu\} F_{\mu\nu} = 0} \qquad (4.60)$$

with $F^{bc}_{\mu\nu} = F^a_{\mu\nu} (t^a)^{bc}$ (adjoint rep.).

In $U(1)$ we have $\partial_\mu \partial_\nu F_{\mu\nu} = 0$.

Now consider the generating functional in eq. (4.46). The ghost fields transform as

$$c \to c^U = U \cdot c\, U^{-1}, \bar{c} \to \bar{c}^U = U \bar{c} U^{-1} \quad (4.61)$$

Infinitesimally

$$c \to c + i[\omega, c], \quad \bar{c} \to \bar{c} + i[\omega, \bar{c}] \quad (4.62)$$

$\Rightarrow$ Generator equivalent to (4.58)

$$S_{gh}[c + i[\omega, c], \bar{c} + i[\omega, \bar{c}], A + \frac{1}{g} D\omega] + O(\omega^2) \quad (4.63)$$

$$= S_{gh}[c, \bar{c}, A] - \left\{ \int_x \frac{1}{g}\, \omega \cdot D \cdot \frac{\delta}{\delta A} - i \int_x \left([\omega, c] \cdot \frac{\delta}{\delta c} - [\omega, \bar{c}] \cdot \frac{\delta}{\delta \bar{c}} \right) \right\} S_{gh}[c, \bar{c}, A]$$

We conclude, that the generator of gauge transformations

$$A \to A^u, \; c \to c^u, \; \bar{c} \to \bar{c}^u$$

is given by

$$\boxed{\mathcal{G}^a = \mathcal{D}_\nu^{ab} \frac{\delta}{\delta A_\nu^b} + g f^{abd} \left[c^b \frac{\delta}{\delta c^d} + \bar{c}^b \frac{\delta}{\delta \bar{c}^d} \right]} \quad (4.63)$$

It follows directly from (4.63) that

$$-\int d^d x \; \omega^b(x) \mathcal{G}^b(x) A_\nu(y) = \mathcal{D}_\nu \cdot \omega(y)$$

$$-\int d^d x \; \omega^b(x) \mathcal{G}^b(x) \; c(y) = i g [\omega(y), c(y)] \quad (4.64)$$

$$-\int d^d x \; \omega^b(x) \; \mathcal{G}^b(x) \; \bar{c}(y) = i g [\omega(y), \bar{c}(y)]$$

and hence

$$S[A + \tfrac{1}{g} \mathcal{D}\omega, \; c + i[\omega, c], \; \bar{c} + i[\omega, \bar{c}]] - S[A, c, \bar{c}]$$
$$= -\frac{1}{g} \int d^d x \, \omega^a \mathcal{G}^a \; S[A, c, \bar{c}] \quad (4.65)$$

In the generating functional this leads to

$$Z[J,\eta,\bar\eta] = \int \mathcal{D}A\,\mathcal{D}c\,\mathcal{D}\bar c\; e^{-S[A,c,\bar c] + \int_x (J\cdot A + \bar\eta\cdot c - \bar c\,\eta)}$$

relabel

$$\underset{(A,c,\bar c)\to(A^u,c^u,\bar c^u)}{=} \int \mathcal{D}A^u\,\mathcal{D}c^u\,\mathcal{D}\bar c^u\, e^{-S[A^u,c^u,\bar c^u] + \int_x (J\cdot A^u + \bar\eta\cdot c^u - \bar c^u\cdot\eta)}$$

$$= \int \mathcal{D}A\,\mathcal{D}c\,\mathcal{D}\bar c\; e^{-S[A^u,c^u,\bar c^u] + \int_x (J\cdot A^u + \bar\eta\, c^u - \bar c^u\,\eta)} \qquad (4.66)$$

and hence infinitesimally

$$\left.\frac{\delta Z[J,\eta,\bar\eta]}{\delta\omega}\right|_{\omega=0} = 0 \qquad (4.67)$$

With eq. (4.65) we arrive at

$$\boxed{\begin{aligned} \langle \mathcal{G}^a(x)\, S[A,c,\bar c]\rangle &= \left\langle \mathcal{D}_\nu^{ab} J_\nu^b(x) + g f^{abd}(\bar\eta^d c^b - \bar c^b \eta^d)\right\rangle \\ &= \bar{\mathcal{D}}_\nu^{ab} J_\nu^b + g f^{abd}(\bar\eta^d \mathcal{C}^b - \bar{\mathcal{C}}^b \eta^d) \qquad (4.68) \\ \text{with} \quad & \bar A = \langle A\rangle \\ & \mathcal{C} = \langle c\rangle, \quad \bar{\mathcal{C}} = \langle \bar c\rangle \end{aligned}}$$

Slavnov-Taylor identity (STI)

Remarks:

(1) Eq. (4.68) also follows as the DSE from translation invariance of the path integral (see p. 60, eqs. (3.5)–(3.9)):

$$\int \mathcal{D}A\,\mathcal{D}c\,\mathcal{D}\bar{c}\left[\left(\frac{\delta}{\delta A_\nu^b}\mathcal{D}_\nu^{ab} + g f^{abd}\left(\frac{\delta}{\delta c^d}c^b + \frac{\delta}{\delta \bar{c}^d}\bar{c}^d\right)\right)e^{-S+\int_x (J\cdot A+\ldots)}\right]$$

$$= 0 \quad (4.69)$$

(2) The expectation value $\langle \mathcal{G}\cdot S\rangle$ contains, like the DSE for scalar theories, loop terms in full props. & vertices. Note also that

$$\langle \mathcal{G}\cdot S\rangle = \langle \mathcal{G}\, S_{gh}\rangle + \langle \mathcal{G}\, S_{gf}\rangle \quad (4.70)$$

(3) Repeating the above analysis for the "ill-defined" generating functional without gauge-fixing we arrive at $\bar{D}_\nu \cdot J_\nu = 0$ cov. conserved current

(4.71)

In summary we had to introduce the gauge fixing for removing the gauge redundancy. The STI, eq. (4.68), encodes the information, how the theory reacts to a gauge transformation. Due to the gauge fixing, the current J_μ^a is <u>not</u> covariantly conserved.

However, if we accompany the gauge transformation with a related change of the gauge fixing condition $\mathcal{F}$ such that $A \to A^u$, $\mathcal{F} \to \mathcal{F}^u$

with $\mathcal{F}^u(\bar{A}^u)=0$ for $\mathcal{F}(\bar{A})=0$, the path integral should be invariant!

Note that such a procedure does <u>not</u> change the FP-operator for linear gauges. This idea is at the root of the BRST-symmetry (Becchi, Rouet, Stora '76, Tyutin '75)
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For shifting our gauge fixing condition we rewrite our gauge-fixed action, eq. (4.40) p. 79, as

$$S[A,c,\bar{c},b] = S_{YM}[A] + S_{gh}[c,\bar{c},A] + S_{gf}[A,b] \tag{4.72}$$

with $$S_{gf}[A,b] = \int d^dx \left\{ -\frac{\xi}{2}\, b^a b^a + b^a \partial_\mu A_\mu^a \right\} \tag{4.73}$$

Evaluating $S[A,c,\bar{c},b]$ on the EoM of b leads to the gauge-fixed action $S[A,c,\bar{c}]$:

$$\left.\frac{\delta S[A,c,\bar{c},b]}{\delta b}\right|_{\bar{b}} = 0 \;:\; \bar{b} = \frac{1}{\xi}\,\partial_\mu A_\mu$$

$$\Rightarrow S_{gf}[A,\bar{b}] = \frac{1}{2\xi}\int_x \left(\partial_\mu A_\mu^a\right)^2 \tag{4.74}$$

This analysis also applies to a Gaussian integration over b: (Exercise)

$$\int \mathcal{D}b\, e^{-S_{gf}[A,b]} \simeq e^{-S_{gf}[A]} \tag{4.75}$$

BRST transformation:

We first notice that under a gauge trafo

$A_\mu \to A_\mu + D_\mu \omega$ we have

absorb $1/g$ in ω

$$b^a \partial_\mu A^a_\mu \to b^a \partial_\mu A^a_\mu + b^a \partial_\mu D^{ab}_\mu \omega^b \tag{4.76}$$

The shifted term is proportional to the FP-operator.

If $\omega^b = \varepsilon c^b$ we can absorb this change

Grassmann-valued

with $\bar{c} \xrightarrow{\delta} \varepsilon b$. It follows that

$$b \cdot \partial_\mu A_\mu - \bar{c}\, \partial_\mu D_\mu c \xrightarrow[\delta:\ \bar{c} \to \bar{c} + \varepsilon b]{\delta:\ A \to A + \varepsilon Dc} b \cdot \partial_\mu A_\mu - \bar{c}\, \partial_\mu D_\mu c \tag{4.77}$$
$$- \bar{c}\, \partial_\mu \big(D_\mu (A + \varepsilon Dc) - D_\mu(A) \big) c$$

with $D_\mu (A + \varepsilon Dc) - D_\mu = + ig [D_\mu \varepsilon c, \cdot\,]$

This entails that

$$ig [D_\mu \varepsilon c, c] + D_\mu \delta c \overset{!}{=} 0 \tag{4.78}$$

$$\parallel \qquad c \varepsilon c = - \varepsilon c^2$$

$$ig \big[(D_\mu \varepsilon c) c + \varepsilon c D_\mu c \big] = ig\, D_\mu (\varepsilon c^2)$$

Note that

$$c^2 = c^a t^a c^b t^b = c^a c^b \frac{1}{2}[t^a, t^b]$$
$$= \frac{1}{2} i f^{abc} c^a c^b t^c \neq 0 \qquad (4.79)$$

We conclude that eq. (4.78) is satisfied if

$$\delta c = i g c^2 \qquad (4.80)$$

We summarise the BRST-transformations

gauge trafo: $\delta_\varepsilon A = \varepsilon D c$

coord. rot. $\delta_\varepsilon c = \varepsilon i g c^2 \qquad (4.81)$

shift of g.f. $\delta_\varepsilon \bar{c} = \varepsilon b$

$\delta b = 0$

which leave the gauge-fixed action invariant:

$$\delta_\varepsilon S[A, c, \bar{c}, b] = 0 \qquad (4.82)$$

as

$$\delta_\varepsilon S_{YM}[A] = \delta_\varepsilon \left(S_{gf}[A, b] + S_{gh}[c, \bar{c}, A] \right) = 0 \qquad (4.83)$$

How does (4.81) generalise for $\partial_\mu A_\mu \to \mathcal{F}(A)$?

Construction of Hilbert space:

In QED we have split the Fock space $\mathcal{F}$ in physical (transversal) polarisations with creation ops. $\alpha^+_{1/2}$, zero-norm states related to α^+_ℓ, and negative norm states created by α^+_-. Polarisations referred to the momentum vector k_ν, see chapter 5.2, p. 137 - 147.

In analogy, we define our Hilbert space states with $\delta_\varepsilon |\psi\rangle =: \varepsilon Q |\psi\rangle = 0$

BRST-operator

The BRST-operator Q is Grassmann-valued. Indeed, it increases the number of ghosts by one, see eq. (4.81).

Remark: $Q\,\hat{O}[A] = 0$ & $Q\,S_{YM}[A] = 0$

relates Q to gauge trafos

We define our physical sub space $\mathcal{F}_{phys}$ by

$$\mathcal{F}_{phys} = \{ |\psi\rangle \in \mathcal{F} \mid Q|\psi\rangle = 0 \} \qquad (4.84)$$
$$= \text{Ker}\, Q$$

This is a linear sub-space of $\mathcal{F}$ (as Q is linear).

$\mathcal{F}_{phys}$ contains zero-norm states: $|\psi_0\rangle = Q|\psi\rangle$

This follows from

$$\boxed{Q^2 = 0} \qquad (4.85)$$

Proof:

$$\text{(a)}\quad Q^2 A = Q\, \mathcal{D} c = -ig\{\overset{\substack{Q \text{ Grassmann-valued} \\ \downarrow}}{QA}, c\} + \mathcal{D} Q c$$
$$= 0$$

$$\text{(b)}\quad Q^2 c = igQ\, c^2 = -g^2 c^2 c \overset{\substack{Q \text{ Grassmann-valued} \\ \downarrow}}{+} g^2 c c^2 = 0$$

$$\text{(c)}\quad Q^2 \bar{c} = Q b = 0 \qquad (4.86)$$

$$\text{(d)}\quad Q^2 b = 0$$

Q is a derivative (defined by eq. (4.85))

With eq. (4.85) we deduce

$$\langle \Psi_0 | \Psi_0 \rangle \underset{\substack{\uparrow \\ Q \text{ hermitian}}}{=} \langle \Psi | Q^2 | \Psi \rangle = 0 \qquad (4.87)$$

$\Rightarrow$ Physical Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \mathcal{F}_{phys} / \sim \qquad (4.88)$$

with $\sim$: $|\Psi_1\rangle \sim |\Psi_2\rangle$ if $Q(|\Psi_1\rangle - |\Psi_2\rangle) = 0$, see chapt. 5.2, QFT I.

This can be rewritten as

$$\boxed{\mathcal{H}_{phys} = \mathrm{Ker}\, Q / \mathrm{Im}\, Q,} \qquad (4.89)$$

(cohomological construction, think of de Rahm cohomology with $d = dx_\mu \frac{\partial}{\partial x_\mu}$ and $d^2 = dx_\mu \wedge dx_\nu \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\mu} = 0$)

Remarks:

(i) $\mathcal{H}_{phys}$ does not include anti-ghosts ($Q\bar{c} = b$)

(ii) b and $\partial_\mu A_\mu$ are equivalent on the EoM of b

(iii) $\partial_\nu A_\nu$ & c terms allowed, but are always equivalent to elements of Im Q

(iv) Explicit construction of $\mathcal{H}_{phys}$ in analogy to Gupta-Bleuler: Kugo-Ojima

Key properties: (a) $[Q, H] = 0$

$\Rightarrow$ asymptotic states at $T \to -\infty$

(b) Q globally defined

$\boxed{\text{assumption in KO}}$

topic of current debate

(v) In QED BRST is unnecessary, but shows nicely the difference between $U(1)$ and $SU(N)$, see e.g. Alkofer, von Smekal Phys. Rep.