4.4 Quantum master equation*

The generating functional shows now BRST invariance,

$$Z[J,\eta,\bar\eta] = \int \mathcal{D}A\,\mathcal{D}C\,\mathcal{D}\bar C\,\mathcal{D}b\, e^{-S[A,C,\bar C,b]} \cdot e^{\int_x (J\cdot A + \bar\eta\cdot C - \bar C\eta)}$$

$$= \int \mathcal{D}A\,\mathcal{D}C\,\mathcal{D}\bar C\,\mathcal{D}b\, e^{-S[A,C,\bar C,b]} \cdot e^{\int_x \{J\cdot(A+\varepsilon QA) + \bar\eta(C+\varepsilon QC) - (\bar C+\varepsilon Q\bar C)\eta\}} \qquad (4.92)$$

$$\Rightarrow \boxed{\int_x \{ J\cdot\langle \varepsilon Q A\rangle + \bar\eta\langle \varepsilon Q C\rangle - \langle \varepsilon Q \bar C\rangle\eta \} = 0} \qquad (4.93)$$

Again this is seemingly a complicated loop equation, as the expectation values are non-trivial. However, the derivative property of Q, $Q^2=0$ in eq. (4.85) comes to our aid:

We add source terms for $QA, QC, Q\bar C$ in Z!

They do not transform under Q due to $Q^2=0$.

We then have the generating functional

$$Z[J,\eta,\bar\eta,L_A,L_C,L_{\bar C}] \tag{4.94}$$
$$= \int \mathcal{D}A\,\mathcal{D}C\,\mathcal{D}\bar C\,\mathcal{D}b\, e^{-S+\int_x \{J\cdot A + \bar\eta C - \bar C \eta + L_A\cdot QA + L_C QC + L_{\bar C} Q\bar C\}}$$

$Q\bar C = +b$: Source term for b

and eq. (4.93) still holds. The expectation values can be represented as

$$\langle QA\rangle = \frac{1}{Z}\frac{\delta Z}{\delta L_A}, \quad \langle QC\rangle = \frac{1}{Z}\frac{\delta Z}{\delta L_C}$$
$$\langle Q\bar C\rangle = \frac{1}{Z}\frac{\delta Z}{\delta L_{\bar C}} \left(=\langle b\rangle\right) \tag{4.95}$$

and we get for eq. (4.93):

$$\boxed{\int_x \left\{ J\circ\frac{\delta Z}{\delta L_A} - \bar\eta\circ\frac{\delta Z}{\delta L_C} - \frac{\delta Z}{\delta L_{\bar C}}\circ\eta \right\} = 0} \tag{4.96}$$

In (4.96) we have used that $\bar\eta\,\varepsilon = -\varepsilon\,\bar\eta$, see eq. (4.93). Note also that $\frac{\delta Z}{\delta L_{\bar C}} = \langle b\rangle$. For $L_{\bar C}=0$ this is simply $\langle b\rangle \underset{\substack{\uparrow \\ \text{Lorentz gauge}}}{=} \partial_\mu \bar A$ with $\bar A = \langle A\rangle$.

Finally we define the effective action, that generates 1PI Green fcts.: (see chapter 3.1, p57.)

$$\boxed{\Gamma[A, c, \bar{c}; L_A, L_c, L_{\bar{c}}] = \left\{\int d^dx \; J\cdot A + \bar{\eta}\cdot c - \bar{c}\cdot\eta - \ln Z[J, \eta, \bar{\eta}, L_A, L_c, L_{\bar{c}}]\right\}} \qquad (4.97)$$

with

$$J = \frac{\delta\Gamma}{\delta A}, \quad \bar{\eta} = -\frac{\delta\Gamma}{\delta c}, \quad \eta = -\frac{\delta\Gamma}{\delta\bar{c}} \qquad (4.98)$$

This leads to the master equation

Grassmann

$$\boxed{\int d^dx \left\{\frac{\delta\Gamma}{\delta L_A}\cdot\frac{\delta\Gamma}{\delta A} + \frac{\delta\Gamma}{\delta L_c}\cdot\frac{\delta\Gamma}{\delta c} + \frac{\delta\Gamma}{\delta L_{\bar{c}}}\cdot\frac{\delta\Gamma}{\delta\bar{c}}\right\} = 0} \qquad (4.99)$$

non-Grassmann

which encodes BRST-invariance of the effective action.

Remarks & examples:

(i) Classically, the effective action is the classical action, eq. (4.72),

$$S[A,C,\bar{C},b;L_A,L_C,L_{\bar{C}}] = S_{YM}[A] + S_{gh}[C,\bar{C},A] + S_{gf}[A,b]$$
$$- \int_x \{ L_A \cdot QA + L_C \cdot QC + L_{\bar{C}} Q\bar{C} \} \qquad (4.100)$$

It follows that

$$\frac{\delta S}{\delta L_A} = -QA, \quad \frac{\delta S}{\delta L_C} = -QC, \quad \frac{\delta S}{\delta L_{\bar{C}}} = -Q\bar{C} \qquad (4.101)$$

and hence:

$$(\Sigma) \int_x \left\{ QA \cdot \frac{\delta S}{\delta A} + QC \frac{\delta S}{\delta C} + Q\bar{C} \frac{\delta S}{\delta \bar{C}} \right\} = 0 \qquad (4.102)$$

which can be rewritten as $\boxed{\delta_\Sigma S = 0}$.

Hence, eq. (4.99) implies classical BRST invariance.

(ii) The generator of quantum BRST-transformations is

$$S_\Gamma := \int_x \left\{ \frac{\delta \Gamma}{\delta L_A} \cdot \frac{\delta}{\delta A} + \frac{\delta \Gamma}{\delta L_C} \cdot \frac{\delta}{\delta C} + \frac{\delta \Gamma}{\delta L_{\bar{C}}} \frac{\delta}{\delta \bar{C}} \right\} \qquad (4.103)$$

(iii) Quantum action principle:

(1) classical: $\partial_\varepsilon S = 0$

(2) 1-Loop: $$\frac{\delta S}{\delta L_A} \circ \frac{\delta \Gamma}{\delta A}\Big|_{1\text{-Loop}} + \frac{\delta \Gamma}{\delta L_A}\Big|_{1\text{-Loop}} \cdot \frac{\delta S}{\delta A} + \dots = 0$$ (4.104)

$\vdots$

n-Loop: $$\frac{\delta \Gamma}{\delta L_A}\Big|_{n-1\text{-Loop}} \circ \frac{\delta \Gamma}{\delta A}\Big|_{n\text{-Loop}} + \frac{\delta \Gamma}{\delta L_A}\Big|_{n\text{-Loop}} \cdot \frac{\delta \Gamma}{\delta A}\Big|_{n-1\text{-Loop}} + \dots = 0$$

(iv) Integrating out b:

$$\frac{\delta \Gamma}{\delta L_{\bar{c}}}\Big|_{L_{\bar{c}}=0} = -\langle b \rangle_{L_{\bar{c}}=0} = -\frac{1}{\xi}\partial_\nu A_\nu$$

$$\Rightarrow \boxed{\int_x \left\{ \frac{\delta \Gamma}{\delta L_A} \circ \frac{\delta \Gamma}{\delta A} + \frac{\delta \Gamma}{\delta L_c}\frac{\delta \Gamma}{\delta C} - \frac{1}{\xi}\partial_\nu A_\nu \cdot \frac{\delta \Gamma}{\delta \bar{C}} = 0 \right.} \quad (4.105a)$$

The anti-ghost only appears linearly in the generating functional Z. Due to translation invariance (DSE) it follows that ($\int DA\, DC\, D\bar{C}\, Db \frac{\delta}{\delta \bar{c}} \dots = 0$)

$$\langle \partial_\nu D_\nu C \rangle + \eta = 0$$

or in terms of Γ: $$\boxed{\partial_\nu \frac{\delta \Gamma}{\delta L_{A_\nu}} - \frac{\delta \Gamma}{\delta \bar{C}} = 0} \quad (4.105b)$$

Putting eq. (4.105a) and eq. (4.105b) together we arrive at

$$\boxed{\int_x \left\{ \frac{\delta\Gamma}{\delta L_A} \cdot \frac{\delta\Gamma}{\delta A} + \frac{\delta\Gamma}{\delta L_c} \circ \frac{\delta\Gamma}{\delta C} - \frac{1}{\xi}\, \partial_\mu A_\mu\, \partial_\nu \frac{\delta\Gamma}{\delta L_{A\nu}} \right\} = 0} \qquad (4.105c)$$

In eq. (4.105c) the b-field is integrated out evidently the variation of the anti-ghost simply amounts to a gauge transformation of the gauge fixing term (as it was introduced in eq. (4.77) in the first place.

(v) Transversality: consider the (amputated) 2-point correlation function $\frac{\delta^2\Gamma}{\delta A^2}$.

We take a $\frac{\delta}{\delta A^a_\mu(x)}\frac{\delta}{\delta C^b(y)}$ - derivative of (4.99) (or (4.105a) at vanishing fields and BRST-sources:

$$\boxed{\int_z \left\{ \frac{\delta^2\Gamma}{\delta C^b(y)\,\delta L^d_{A_\nu}(z)} \cdot \frac{\delta^2\Gamma}{\delta A^d_\nu(z)\,\delta A^a_\mu(x)} + \frac{\delta^2\Gamma}{\delta L^d_{\bar c}(z)\,\delta A^a_\mu(x)}\,\frac{\delta^2\Gamma}{\delta C^b(y)\,\delta \bar C^d(z)}\right\} = 0 \qquad (4.106)}$$

with $\left(\frac{\delta^2\Gamma}{\delta L_{\bar c}\,\delta A_\mu}\right)^{da}_\mu (z,x) \overset{\downarrow}{=} -\frac{1}{\xi}\,\frac{\delta\, \partial_\nu A^d_\nu(z)}{\delta A^a_\mu(x)}$. (Integrating out b, see p. 102, 102a)

$$\left(\frac{\delta^2\Gamma}{\delta c\,\delta \bar c}\right)^{bd}_{(y,z)} = \frac{\delta}{\delta C^b(y)}\,\partial_\nu\,\frac{\delta\Gamma}{\delta L^d_{A_\nu}(z)} \qquad (4.107)$$

eq. (4.105b)

$$= \partial^z_\nu\,\frac{\delta^2\Gamma}{\delta C^b(y)\,\delta L^d_{A_\nu}}$$

In summary:

$$\boxed{\int_z \frac{\delta^2\Gamma}{\delta L^d_{A_\nu}(z)\,\delta C^b(y)}\left\{\underbrace{\frac{\delta^2\Gamma}{\delta A^d_\nu(z)\,\delta A^a_\mu(x)} + \frac{1}{\xi}\,\partial_\mu\partial_\nu\,\delta^d(z-x)}_{-\frac{\delta^2 S_{gf}[A]}{\delta A^d_\nu(z)\,\delta A^a_\mu(x)}}\right\} = 0}$$

with $\left(\frac{\delta^2\Gamma}{\delta L_A\,\delta C}\right)^{db}_\nu (z,y) = \partial_\nu\,\delta^d(z-y)\,\delta^{db} + (1\text{-loop}) + \ldots$ (4.108)