5.2 Running coupling

for QED see QFT I, p. 195-204

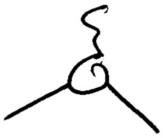
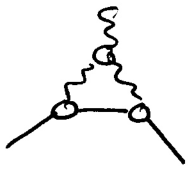
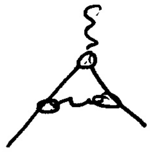
The renormalised coupling

$$g = g_0 / Z_g \qquad (5.21)$$

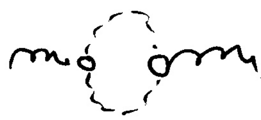
is computed e.g. from the ratio

$$Z_g = Z_{1,F} \Big/ Z_2 \, Z_3^{1/2} \qquad (5.22)$$

that is from the diagrams (at 1-loop)

 $\Big|_{1-\mathrm{loop}}$ =  + 

 $\Big|_{1-\mathrm{loop}}$ = 

 $\Big|_{1-\mathrm{loop}}$ =  +  + 

+ 

In dim. reg. this leads to diagrams with

$$\mu^{2\varepsilon} \int \frac{d^d l}{(2\pi)^d} \Big[\text{loop integrand} \Big]$$

that is, $Z = Z(\mu)$.

The bare parameters do <u>not</u> depend on μ, and hence

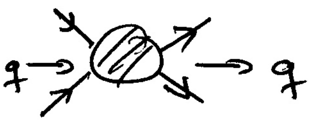
$$\boxed{\beta_g = \mu \frac{d}{d\mu}\, g = - g\, \mu \frac{d}{d\mu} \ln Z_g} \qquad (5.21)$$

However, Observables $\mathcal{O}$ do also <u>not</u> depend on μ:

$$\boxed{\mu \frac{d}{d\mu}\, \mathcal{O} = 0} \qquad (5.22)$$

Apply Eq. (5.22) to a cross-section, e.g.

$$\mathcal{O}\left(g^2, q^2/\mu^2 \right) =$$



No dep. on quark mass: massless quarks
or heavy quark limit: $m \to \infty$
(quenched)

and with eq. (5.22) we have

$$\mu \frac{d}{d\mu} \mathcal{O} = \boxed{\left(\mu \frac{\partial}{\partial\mu} + \beta\, \partial_g \right) \mathcal{O} = 0} \qquad (5.23)$$

RG-equation

Now, if we choose $\mu^2 = q^2$:

$$\sigma(g^2, q^2/\mu^2) = \sigma(4\pi\alpha_s, 1) \qquad (5.24)$$

with running coupling

$$\boxed{\alpha_s(q^2) = \frac{g^2(q^2)}{4\pi}} \qquad (5.25)$$

Remark:

(1) In QFT I we have seen that we could pick the renormalisation condition $g^2\big|_{q^2=\mu^2} = g^2_{phys}$

By choosing $\mu^2 = q^2$ this is then valid at all scales.

(2) The sign of the β-function in eq. (5.21) then tells us, whether the interaction strength grows ($\beta > 0$) or gets smaller ($\beta < 0$) for larger momenta.

Rehash dimensional regularisation: e.g. QFT I, p. 187

$$I_{\mu_1\mu_2} = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^{2n}} \frac{1}{(l+p)^{2m}} T_{\mu_1\mu_2}(p,l)$$

with $I_{\mu_1\mu_2}(p,l) = \{ \delta_{\mu_1\mu_2}, p_{\mu_1}p_{\mu_2}, l_{\mu_1}l_{\mu_2}, p_{\mu_1}l_{\mu_2} \}$

$$I_{\mu_1\mu_2} \longrightarrow \mu^{2\varepsilon} \int \frac{d^d l}{(4\pi^2)^{d/2}} \frac{1}{l^{2n}} \frac{1}{(l+p)^{2m}} T_{\mu_1\mu_2}(p,l)$$

with $d = 4 - 2\varepsilon$

Remarks:

(i) $\left[I_{\mu_1\mu_2} \right]_{d=4} = \left[I_{\mu_1\mu_2} \right]_{d=4-2\varepsilon}$

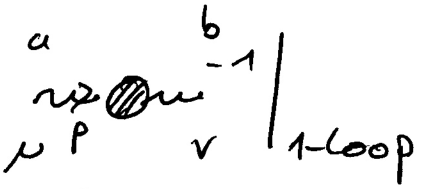
since $\left[\mu^{2\varepsilon} d^d l \right] = 4 \quad \forall d$

(ii) dependence on new scale μ.

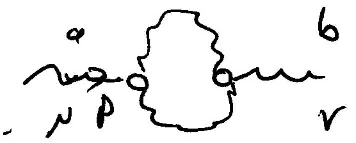
For the running coupling we need e.g. $Z_{1,F}$ (quark self-energy), Z_2 (quark-gluon vertex) and Z_3 (gluon vac. pol.).

<u>One loop computation:</u> (explicit in Feynman (Feynman rules, p. 82, 107) gauge)

Vacuum polarisation:



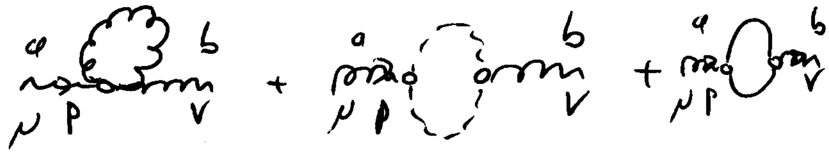
$\simeq$ ⓐ



$\Vert$

$$\left(p^2\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$$

$+$



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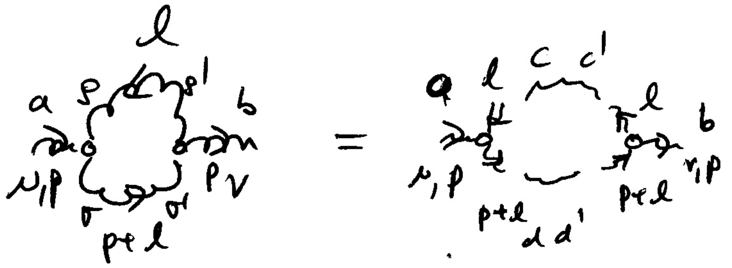


$\leftarrow$ Feynman rules p. 82

$$ⓐ = \frac{1}{2}\underbrace{g^2\mu^{2\varepsilon}}_{g \text{ in } d \text{ dim}}\int\frac{d^dl}{(2\pi)^d}\, f^{acd}\,\big[\delta_{\mu\rho}(p-l)_\sigma + \delta_{\rho\sigma}(2l+p)_\mu + \delta_{\sigma\mu}(-2p-l)_\rho\big]\,\frac{\delta_{\rho\rho'}}{l^2}\, f^{bcd}\big[\delta_{\nu\rho'}(-p+l)_{\sigma'} + \delta_{\rho'\sigma'}(-2l-p)_\nu + \delta_{\sigma'\nu}(2p+l)_{\rho'}\big]\frac{\delta_{\sigma\sigma'}}{(l+p)^2}\Big\} \quad (5.26)$$

Traces & contractions:

$$f^{bdc} = -f^{bcd}$$

$$f^{acd} f^{bcd} = (-i f^a)^{cd} (-i f^b)^{dc}$$

$$= \mathrm{tr}\, t^a t^b \quad \text{with } (t^a)^{bc} = -i f^{abc} \tag{5.27}$$

t^a are generators in the adjoint ($N^2-1 \times N^2-1$ dim) representation of the gauge group.

[The t^a satisfy the Lie-algebra (see exercise sheet) as $f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$.

In general we have

$$\underbrace{\mathrm{Tr}_R}_{\text{Representation}}\, t^a t^b = \overbrace{C(R)}^{\text{Dynkin index}} \delta^{ab} \tag{5.28}$$

$\boxed{C(\text{fund}) = 1/2}$

The second Casimir operator of a group is defined by

$$t_R^a t_R^a = C_2(R)\, \mathbb{1}_R \tag{5.29}$$

(e.g. angular momentum squared L^2 in QM)

For the adjoint rep. we have

$$C_2(\text{ad}) = C(\text{ad}) = N \tag{5.30}$$

$$\Rightarrow (a) = \frac{1}{2} g^2 \mu^{2\varepsilon} N \int \frac{d^d l}{(2\pi)^d} \left[\delta_{\nu\rho} \cdots \right] \left[\delta_{\nu\rho} \cdots \right] \frac{1}{l^2} \frac{1}{(l+p)^2} \tag{5.31}$$

We use the Feynman trick (QFT I, eq. 7.56, p.2?? and sheet 11)

$$\frac{1}{l^2} \frac{1}{(l+p)^2} = \int_0^1 d\alpha \, \frac{1}{\alpha (l+p)^2 + (1-\alpha)\, l^2}$$

$$= \int_0^1 d\alpha \, \frac{1}{(k^2 + \Delta)^2} \tag{5.32}$$

with $k = l + \alpha p$ and $\Delta = \alpha(1-\alpha)\, p^2$.

We can also shift the l-integration: $\int d^d l = \int d^d k$.

It is left to rewrite the numerator in eq. (5.31) (see (5.26)) in terms of k: $l = k - \alpha p$.

(i) terms linear in k_μ integrate to zero:

$$\int d^d k \, \frac{1}{(k^2+\Delta^2)^2} \, k_\mu = 0 \tag{5.33a}$$

(ii) terms with $k_\mu k_\nu$ are prop. to $\delta_{\mu\nu}$:

$$\int d^d k \, \frac{1}{(k^2+\Delta^2)^2} k_\mu k_\nu = \frac{1}{d} \int d^d k \, \frac{k^2}{(k^2+\Delta^2)^2} \tag{5.33b}$$

In summary:

$$ⓐ = \delta^{ab} N g^2 \int_0^1 d\alpha \, \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+\Delta)^2} \frac{N_{\mu\nu}}{2} \qquad (5.34)$$

with $\quad N_{\mu\nu} = [\, \delta_{\mu\rho}(p-\ell)_\sigma + \delta_{\rho\sigma}(2\ell+p)_\mu + \delta_{\mu\sigma}(-2p-\ell)_\rho]$

$$\cdot [\, \delta_{\nu\rho}(-p+\ell)_\sigma + \delta_{\rho\sigma}(-2\ell-p)_\nu + \delta_{\sigma\nu}(2p+\ell)_\rho]$$

$\ell = k - \alpha p \rightarrow \quad = [\, \delta_{\mu\rho}(p(1+\alpha)-k)_\sigma + \delta_{\rho\sigma}(2k+p(1-2\alpha))_\mu + \delta_{\mu\sigma}(-k-(2-\alpha)p)_\rho]$

$$\cdot [\, \delta_{\nu\rho}(-(1+\alpha)p+k)_\sigma + \delta_{\rho\sigma}(-2k-p(1-2\alpha))_\nu + \delta_{\nu\sigma}(k+(2-\alpha)p)_\rho]$$

linear terms in k vanish

$$\boxed{\doteq \delta_{\mu\nu}\left[-p^2\left((1+\alpha)^2+(2-\alpha)^2\right) - 2k^2\right] + p_\mu p_\nu\left((2-d)(1-2\alpha)^2+2(1+\alpha)(2-\alpha)\right) + k_\mu k_\nu(6-4d)\,]} \qquad (5.35)$$

+ lin. terms

This amounts to $\left[\int d^d k \, k_\mu k_\nu f(k^2) = \frac{1}{d} \int d^d k \, k^2 f(k^2) \right]$

$$\boxed{ⓐ = \frac{\delta^{ab}}{2} N g^2 \int_0^1 d\alpha \, \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \Big[-\delta_{\mu\nu}\left(p^2\left((1+\alpha)^2+(2-\alpha)^2\right) + 6k^2(1-1/d)\right) + p_\mu p_\nu \left((2-d)(1-2\alpha)^2+2(1+\alpha)(2-\alpha)\right)\Big] \frac{1}{(k^2+\Delta)^2}} \qquad (5.36)$$

Now we use that

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \quad \frac{1}{\Delta^{n-d/2}} \quad \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

(see also sheet 11, QFT I) (5.37)

$$\Rightarrow \boxed{ (a) = \frac{1}{2} \frac{\delta^{ab} N g^2}{(4\pi)^{d/2}} \int_0^1 d\alpha \, \frac{\mu^{2\varepsilon}}{\Delta^{2-d/2}} \Big\{ -\delta_{\mu\nu}\, p^2 \Big[\Gamma(2-d/2)\big((1+\alpha)^2+(2-\alpha)^2\big) + \Gamma(1-d/2)\, 3\, (d-1)\, \alpha(1-\alpha) \Big] + p_\mu p_\nu \, \Gamma(2-d/2) \Big[(2-d)(1-2\alpha)^2 + 2(1+\alpha)(2-\alpha) \Big] \Big\} } \qquad (5.38)$$

from eq. (3.37): $\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2-1)}{\Gamma(n)} \frac{1}{\Delta^{n-d/2-1}} \frac{d}{2}$

Tadpole (b):

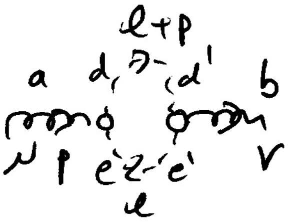
$$= \underbrace{-g^2 N (d-1) \delta^{ab} \delta_{\mu\nu} \mu^{2\varepsilon}}_{\text{see sheet 9}} \underbrace{\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2}}_{0 \text{ in dim-reg}}$$

However, multiplying the integrand with $\frac{(p+l)^2}{(p+l)^2}$ leads to

$$= \frac{\delta^{ab} \delta_{\mu\nu} N g^2}{(4\pi)^{d/2}} \int_0^1 d\alpha \, \frac{\mu^{2\varepsilon}}{\Delta^{2-d/2}} (d-1)\, p^2 \Big\{ \Gamma(1-d/2)\, d\, \alpha(1-\alpha) + \Gamma(2-d/2)\, (1-\alpha)^2 \Big\}$$

(5.39)

Ghost-contribution (c):

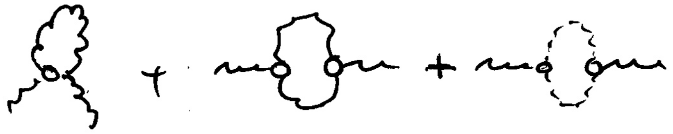


$$= \overset{\text{fermion loop}}{-}\, g^2 \mu^{2\varepsilon} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l+p)^2} \frac{1}{l^2} \underbrace{f^{dae} f^{ebd}}_{-N} \cdot (l+p)_\mu l_\nu \qquad (5.40)$$

$$l = k - \alpha p \;\rightarrow\; = \delta^{ab} g^2 N \int_0^1 d\alpha\, \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+\Delta)^2} \left(k_\mu k_\nu - \alpha(1-\alpha) p_\mu p_\nu \right)$$

$$= \frac{\delta^{ab} N g^2}{(4\pi)^{d/2}} \int_0^1 d\alpha \frac{\mu^{2\varepsilon}}{\Delta^{2-d/2}} \Big\{ \delta_{\mu\nu} p^2 \Gamma(1-d/2) \tfrac{1}{2} \alpha(1-\alpha) - p_\mu p_\nu \Gamma(2-d/2) \alpha(1-\alpha) \Big\} \qquad (5.41)$$

Pure Yang-Mills:



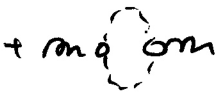
$$\boxed{\;+\;+\; = -\frac{\delta^{ab} N g^2}{(4\pi)^{d/2}} \int_0^1 d\alpha \frac{\mu^{2\varepsilon}}{\Delta^{2-d/2}} \Gamma(2-d/2) \cdot \left(\delta_{\mu\nu} p^2 - p_\mu p_\nu \right) \Big\{ \left(1-\frac{d}{2}\right)(1-2\alpha)^2 + 2 \Big\} \qquad (5.42)}$$

For $\varepsilon \to 0$ the dimension d approaches 4 : $d = 4 - 2\varepsilon$

Then, eq. (3.34) diverges as $1/\varepsilon$ because of $\Gamma[2 - d/2] = \Gamma[\varepsilon]$, see also QFT I, chapter 7, p. 187 :

$$\Gamma[\varepsilon] = \frac{1}{\varepsilon} - \underbrace{\gamma}_{\substack{\text{Euler-Mascheroni} \\ \gamma = 0.577\ldots}} + O(\varepsilon)$$

Hence in this limit we get

 +  + 

$$\boxed{= -\frac{g^2 N}{16\pi^2}\left(p^2 \delta_{\mu\nu} - p_\mu p_\nu\right)\delta^{ab} \cdot \left[\underbrace{\frac{5}{3}}_{\left[\int_0^1 d\alpha\, (1-d/2)(1-2\alpha)^2 + 2\right]}\left(\frac{1}{\varepsilon} - \ln p^2/\mu^2\right) + \text{finite}\right] \qquad (5.43)}$$

$$\frac{\mu^{2\varepsilon}}{(\alpha(1-\alpha)p^2)^\varepsilon} = 1 + \varepsilon \ln \mu^2/p^2 - \varepsilon \ln \alpha(1-\alpha) + O(\varepsilon^2)$$

Remarks: (1) Singular diagrams are always proportional to $(1/\varepsilon - \ln p^2/\mu^2)$.

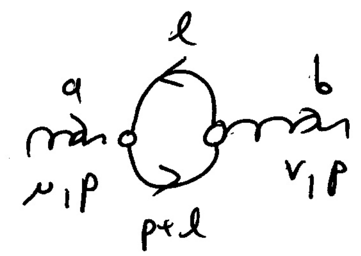
(2) For general $\xi \neq 1$ we get

$$5/3 \to 5/3 + \frac{1}{2}(1-\xi) \qquad (5.44)$$

Exercise!

In the presence of quarks we have to add the diagram (d):

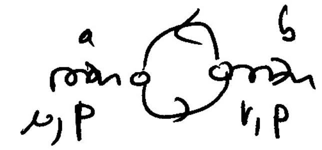
Quark contribution (d):



$$= \overset{\text{fermion loop}}{\underset{\downarrow}{-}}\; g^2 \mu^{2\varepsilon} \int \frac{d^d l}{(2\pi)^d} \overset{i^2}{(-1)}\; \overset{\frac{1}{2}\delta^{ab}}{\overset{\parallel}{\mathrm{tr}_{fund}\, t^a t^b}} \qquad (5.45)$$

$$\mathrm{tr}_{Dirac} \cdot \frac{1}{i\not{l} + m} \gamma_\nu \frac{1}{i(\not{l}+\not{p}) + m} \gamma_\mu$$

[Computation see QFT I, chapter 7.2, p. 200-204]

$\Rightarrow$ 

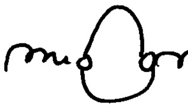
$$= -\frac{1}{2} \frac{g^2}{(4\pi)^{d/2}} \int_0^1 d\alpha \frac{\mu^{2\varepsilon}}{(\Delta + m^2)^{2-d/2}} \frac{4}{d-1} \frac{1}{p^2}$$

$$\Big\{ (d-2)\,\Gamma(1-d/2)(\Delta + m^2) - (d-2)\,\Gamma(2-d/2)(2\Delta + m^2)$$

$$+ d\,\Gamma(2-d/2)\, m^2 \Big\}$$

$$\cdot\, \delta^{ab} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \qquad (5.46)$$

$d \to 4$:



$$= \frac{4g^2}{(4\pi)^2}\int_0^1 d\alpha\,\alpha(1-\alpha)\,\frac{\Gamma(2-d/2)}{(\Delta+m^2)^{\varepsilon}}$$

$$\cdot\,\delta^{ab}\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right) \qquad (5.45)$$

$$+\text{ finite}$$

$$= \frac{g^2 N}{16\pi^2}\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)\delta^{ab}$$

$$\cdot\left[\frac{2}{3}\cdot\left(\frac{1}{\varepsilon}-\int_0^1 d\alpha\,\ln\frac{\Delta+m^2}{\mu^2}\right)+\text{finite}\right]$$

In summary we get for the one-loop correction for

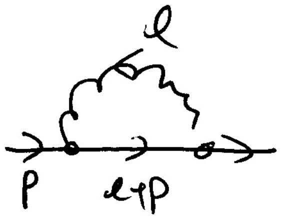
$${}^{-1}\Big|_{1\text{-loop}} = \frac{\delta^2\Gamma}{\delta A^2}:\quad \boxed{m=0}$$

$${}^{-1}\Big|_{1\text{-loop}} = \frac{g^2 N}{16\pi^2}\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)\delta^{ab}$$

$$\cdot\left[\left(\frac{5}{3}+\frac{1}{2}(1-\xi)\right)N-\frac{2}{3}\right]\left(\frac{1}{\varepsilon}-\ln p^2/\mu^2\right)$$

$$+\text{ finite} \qquad (5.46)$$

For the β-function we also need the quark self-energy: $\boxed{\xi = 1}$



$$= -g^2 \mu^{2\varepsilon} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} t^a \gamma_\nu \frac{1}{i(\not{l}+\not{p})} \gamma_\nu t^a$$

$t_f^a t_f^a = C_2(\text{fund})\, 1\!\!1_f$ ⟶

$$= \frac{(N^2-1)}{2N} \frac{g^2}{(4\pi)^{d/2}} i \not{p}\, \mu^{2\varepsilon} \int_0^1 d\alpha\, (1-\alpha)(d-2) \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

p. 117

$C_2(\text{fund}) = \frac{N^2-1}{2N}$

$\mathrm{tr}\, t^a t^a = C_2(R) \cdot \dim(R) = C(R) \cdot (N^2-1)$

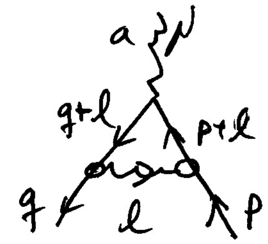
$$= \frac{N^2-1}{2N} \frac{g^2}{(4\pi)^{d/2}} i \not{p} \left[\left(\frac{1}{\varepsilon} - \ln p^2/\mu^2 \right) + \text{finite} \right] \quad (5.47)$$

Gauge dependence: Exercise

$$\left(\frac{1}{\varepsilon} - \ln p^2/\mu^2 \right) \longrightarrow \xi \left(\frac{1}{\varepsilon} - \ln p^2/\mu^2 \right) \quad (5.48)$$

Accordingly, there is no singularity in the self-energy for Landau gauge: $\xi = 0$

Quark - gluon vertex:



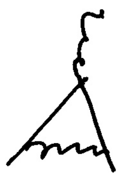
$$= i g^3 \int \frac{d^d l}{(2\pi)^4} \frac{t^b t^a t^b \gamma_\nu (\not l + \not q)\gamma_\mu (\not l + \not p)\gamma_\nu}{l^2 (l+q)^2 (l+p)^2} \qquad (5.49)$$

with $t^b t^a t^b$

$$= t^{b\,2} t^a + t^b [t^a, t^b]$$

$$= C_2(\text{fund}) t^a + i\, t^b f^{abc} t^c$$

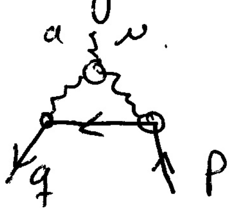
$$= \left[C_2(\text{fund}) - \frac{1}{2} C_2(\text{adj.}) \right] t^a$$

$\Rightarrow$ 

$$\Big|_{q,p=0} \simeq i g t^a \frac{g^2}{(4\pi)^{d/2}} \left(\frac{1}{2} \frac{N^2-1}{N} - \frac{N}{2} \right) \cdot \Big(\underbrace{\Gamma\left(2 - \frac{d}{2}\right)}_{d\to 4:\ 1/\varepsilon} + \text{finite} \Big) \qquad (5.50)$$

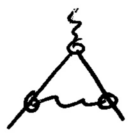
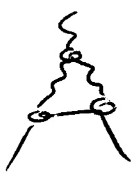
$\int d\to 4$

Similarily:



$$\simeq i g t^a \frac{g^2}{(4\pi)^{d/2}} \frac{3}{2} N \left(\Gamma(2 - d/2) + \text{finite} \right)$$

and hence

 $+$ 

$$\simeq i g t^a \frac{g^2}{(4\pi)^{d/2}} \overbrace{\left(\frac{3}{2} N - \frac{1}{2N} \right)}^{C_2(\text{fund}) + C_2(\text{adj.})} \left(\Gamma(2 - d/2) + \text{finite} \right) \qquad (5.51)$$

β- function: We recall that

$$\beta_g = -g\,\mu \frac{d}{d\mu} \ln Z_g$$

and $\quad Z_g = Z_{1,F} / Z_2 Z_3^{1/2} = \dfrac{Z_{1,F}}{Z_4 Z_A^{1/2}} \qquad (5.52)$

We write $Z = 1 + \delta Z$ ← cancels infinity and get at one loop:

$$Z_g = 1 + \delta Z_{1,F} - \delta Z_4 - \frac{1}{2}\delta Z_A \qquad (5.53)$$

We collect the δZ's:

N_f flavours

$$\delta Z_A \simeq \frac{g^2 N}{16\pi^2}\left[\left(\frac{5}{3} + \frac{1}{2}(1-\xi)\right) - \frac{2}{3}N_f\right]\frac{1}{\varepsilon}$$

$$\delta Z_4 \simeq -\frac{g^2}{16\pi^2}\,\frac{N^2-1}{2N}\,\xi\,\frac{1}{\varepsilon}$$

$$\delta Z_{1,F} \simeq -\frac{g^2}{16\pi^2}\left[\left(1 - \frac{1-\xi}{4}\right)N + \xi\,\frac{N^2-1}{2}\right]\frac{1}{\varepsilon}$$

(5.54)

We also have

$$\delta Z_c = \frac{g^2 N}{16\pi^2}\left[\frac{1}{2} + \frac{1-\xi}{4}\right]\frac{1}{\varepsilon}$$

(5.55)

and

$$\delta \tilde{Z}_1 = -\frac{g^2 N}{16\pi^2}\,\xi/2\,\frac{1}{\varepsilon}$$

$(\bar{c}Ac)$

$\Rightarrow$

$$\beta(g) = -g\,\mu \frac{d}{d\mu} \ln Z_g$$

$$= -g\,\mu \frac{d}{d\mu}\left(\delta Z_{1,F} - \delta Z_\psi - \frac{1}{2}\delta Z_A\right) + O(\text{2-Loop})$$

$\delta Z \sim \left(\frac{1}{\varepsilon} - \ln \mu^2/p^2\right)$

$$= \frac{g^3}{16\pi^2}\,2\Big[\overbrace{-\left(\left(1-\frac{1-\cancel{\xi}}{4}\right)N - \cancel{\xi\frac{N^2-1}{2N}}\right)}^{\delta Z_{1,F}} \qquad (5.56)$$

$$+ \underbrace{\cancel{\frac{N^2-1}{2N}\xi}}_{-\delta Z_\psi} \underbrace{-\frac{1}{2}\left(\left(\frac{5}{3}+\frac{1}{2}(1-\cancel{\xi})\right)N - \frac{2}{3}N_f\right)}_{1/2\,\delta Z_A}\Big]$$

$$\Rightarrow \boxed{\begin{aligned}\beta(g)\Big|_{\text{1-loop}} &= \frac{g^3}{16\pi^2}\Big[-\frac{11}{3}\underset{\substack{\uparrow\\ C_2(adj)}}{N} + \frac{4}{3}N_f\underset{\substack{\uparrow\\ C(fund)}}{\frac{1}{2}}\Big]\\ &= \beta_1 \cdot g\end{aligned}} \qquad (5.57)$$

Remarks:

(1) For $N_f < \frac{22}{4}N$: $\beta < 0$

(2) The β-function can be integrated to obtain

$$\alpha_s(p) = \frac{\alpha_s^0}{1-\beta_1 \ln p^2/\mu_0^2} \qquad (5.58)$$

$p \to \infty$: $\alpha_s(p) \simeq -\frac{\alpha_s^0}{\beta_1 \ln p^2/\mu_0^2} \to 0$ asymptotic freedom