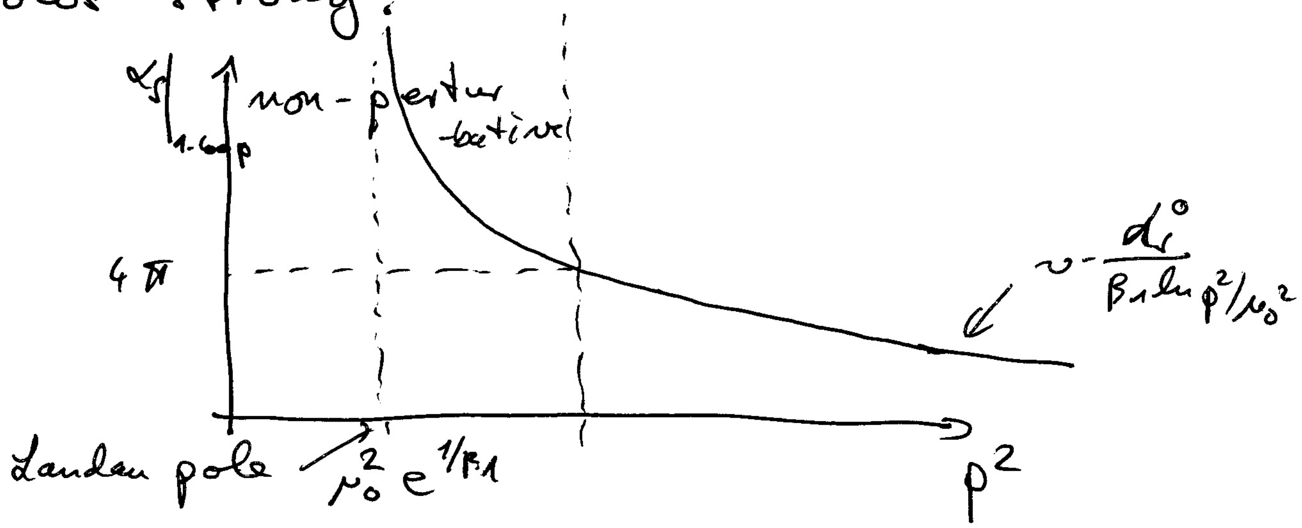
6 Lattice gauge theory

The QCD coupling α_s tends to zero for large momenta, see eq. (5.58), and perturbation theory works.

For momentum p getting small, the coupling grows strong:



Hence, for low momenta <u>non</u>-perturbative methods are required: $\boxed{\text{Numerically solving the path-int.}}$

Remark: Taking into account higher orders in perturbation theory only slightly shifts the Landau pole, and does not remove it.

6.1 Scalar fields on the lattice

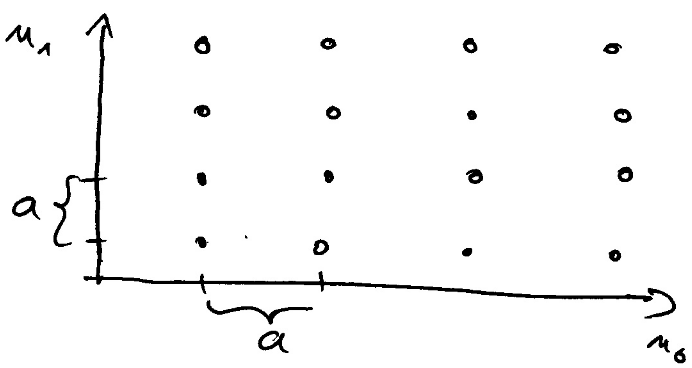
We consider the action of a complex scalar field,

$$S[\phi] = \int d^4x \; \phi^\dagger(x) \left(- \overbrace{\partial_\mu \partial_\mu}^{\Delta} + m^2 \right) \phi(x) \qquad (6.1)$$

In chapter 1.2 we derived the path integral from a discretised integral in time-direction. Now consider a discretized integral in all directions:

$$\mathcal{D}\phi(x) = \prod_{\hat{n}} d\phi(n a) \qquad (6.2)$$

with $n = \begin{pmatrix} n_0 \\ \vdots \\ n_3 \end{pmatrix}$, $n_\mu \in \mathbb{Z}$



with a hypercubic lattice with <u>lattice spacing</u> a.

We have the substitutions

$$x_\mu \to n_\mu a$$

$$\phi(x) \to \frac{1}{a}\hat{\phi}_{\hat{n}}$$

$$\int d^4x \to a^4 \sum_{\hat{n}} \qquad \text{dimensionless} \qquad (6.3)$$

$$\Delta \phi \to \frac{1}{a^2}\hat{\Delta}\phi$$

$$m \to \frac{1}{a}\hat{m}$$

$\hat{\mu} = e_\mu$: unit vector in μ-direct.

with

$$\hat{\Delta}\hat{\phi}_n = \sum_\mu \left[\hat{\phi}_{n+\hat{\mu}} + \hat{\phi}_{n-\hat{\mu}} - 2\hat{\phi}_n \right] \qquad (6.4)$$

discrete Laplacian

Note that $\hat{\Delta}\phi(\hat{n}a) = \sum_\mu \left[\overset{\partial^R_\mu \hat{\phi}_n}{(\hat{\phi}_{n+\hat{\mu}} - \hat{\phi}_n)} - \overset{\partial^L_\mu \hat{\phi}_n}{(\hat{\phi}_n - \hat{\phi}_{n-\hat{\mu}})} \right]$

and $\hat{\partial}_\mu \hat{\phi} = \frac{1}{2}\left(\hat{\phi}_{n+\hat{\mu}} - \hat{\phi}_{n-\hat{\mu}}\right)$ ← symmetric deriv. $\qquad (6.5)$

The derivatives $\frac{1}{a}\hat{\Delta}$, $\frac{1}{a}\hat{\partial}_{L/R_\mu} \xrightarrow{a\to 0} \Delta, \partial_\mu$ in the cont. limit.

In summary, the discretised lattice action of a scalar field takes the form

$$S[\hat{\phi}] = \sum_{n,m} \phi^\dagger_n K_{nm} \hat{\phi}_m \qquad (6.6)$$

with

$$K_{nm} = -\sum_{\mu>0} \left[\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{\hat{n},\hat{m}} \right] + \hat{m}^2 \delta_{nm} \qquad (6.7)$$

The momentum integration in eq. (6.12) is restricted to the <u>Brillouin zone</u> $\hat{p}_\mu \in [-\pi, \pi]$.

Continuum limit:

(i) $$\lim_{a\to 0} \frac{1}{a^2} \tilde{K}(\hat{p}) = \frac{1}{a^2}\left[4\sum_{\mu=0}^{3}\left(\frac{1}{4}(ap_\mu)^2 + \mathcal{O}(a^4)\right) + a^2 m^2\right]$$

$$= p^2 + m^2 \qquad (6.13)$$

(ii) $$\lim_{a\to 0} \frac{1}{a^2} \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4} \frac{e^{i\hat{p}(n-m)}}{\tilde{K}(\hat{p})} = \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(\overset{na}{x} - \overset{m\cdot a}{y})}}{p^2 + m^2} \qquad (6.14)$$

(iii) $S[\hat{\phi}] \to S[\phi]$

Remarks:

(1) The discrete lattice generating functional can be computed numerically:

(a) put the theory on a finite lattice: $|n_\mu a| \leq L$

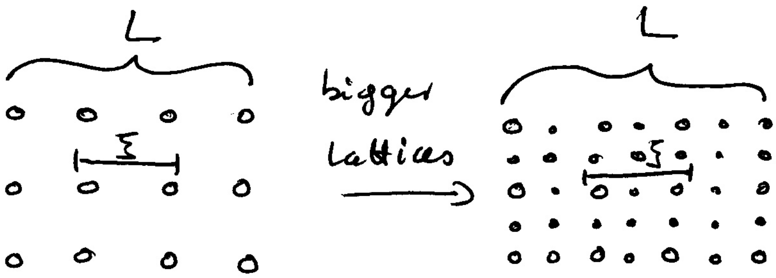
(b) Perform $(L/a)^4$ integrations with Monte-Carlo methods. Common sizes for dynamical QCD simulations $\sim 16^4, 32^4$.

(c) Interaction term

$$S_I(\hat{\phi}) = \frac{\hat{\lambda}}{4!} \sum_n \hat{\phi}^4(\hat{n}a) \qquad (6.15)$$

(2) Continuum limit:

(a)



ξ: correlation length

(b)