

6.2 Non-Abelian gauge fields on the lattice

As in the continuum we introduce

gauge trafo. for the complex matter field $\hat{\Phi}$ (fund. rep.):

$$\hat{\Phi}_n \rightarrow G(n) \hat{\Phi}_n \quad \text{with } G(n) \in SU(N) \quad (6.16)$$

Looking at the action eq. (6.6) on p. 131

we have to ensure gauge invariance for

e.g.
$$\hat{\Phi}_n^\dagger \hat{\Phi}_{n+\hat{\rho}} \quad (6.17)$$

in the action $S[\hat{\Phi}] = \frac{1}{2} \sum_{n,m} \hat{\Phi}_n^\dagger U_{nm} \hat{\Phi}_m$.

This entails that we have to transport the

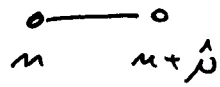
group element $G^+(n+\hat{\rho})$ from the lattice point

$n+\hat{\rho}$ to n : We define the link variable

$$U_\rho(n) : U_\rho(n) \rightarrow G(n) U_\rho(n) G^+(n+\hat{\rho})$$

with parameterisation
$$U_\rho(n) = e^{i \hat{\varphi}^\rho(n)} \quad \begin{matrix} SU(N) \\ \cup \\ SU(N) \end{matrix} \quad (6.18)$$

(1) $U_\nu(n)$ 'lives' on the link between n and $n+\hat{\nu}$:



(2) The term

$$\begin{aligned} \hat{\phi}_n^+ U_\nu(n) \hat{\phi}_{n+\nu} &\rightarrow \hat{\phi}_n^+ G(n)^\dagger G(n) U_\nu(n) G(n+\hat{\nu}) G(n+\hat{\nu})^\dagger \hat{\phi}_{n+\nu} \\ &= \hat{\phi}_n^+ U_\nu(n) \hat{\phi}_{n+\nu} \end{aligned} \quad (6.19)$$

is gauge invariant. This also holds

for $\hat{\phi}_n^+ U_\nu(n+\hat{\nu}) \hat{\phi}_{n+\nu}$. Then the action

turns into (6.20)

$$S[\hat{\phi}, U] = - \sum_{\substack{n \\ \nu > 0}} \left(\hat{\phi}_n^+ U_\nu(n+\hat{\nu}) \hat{\phi}_{n+\nu} + \hat{\phi}_n^+ U_\nu(n) \hat{\phi}_{n+\hat{\nu}} \right) + \sum_n \hat{\phi}_n^+ \hat{\phi}_n (m^2 + \delta)$$

(3) Continuum limit: Write $U_\nu(n) = e^{i \overbrace{g_0}^{\text{bare lattice coupling}} a} A_\nu(n)$ (6.21)

and expand (6.20) in powers of the lattice spacing a .

$$\boxed{\hat{A}_\nu(n) = i g_0 a A_\nu(n)} \quad (6.22)$$

We get

$$S[\hat{\phi}, u] \xrightarrow{a \rightarrow 0}$$

$$O(a^2) + iag_0 \left[\overbrace{A_\nu(n) - A_\nu(n-\hat{\nu})}^{\hat{D}_\nu^L A_\nu(n)} \hat{\phi}_n + A_\nu(n) (\hat{\phi}_{n+\hat{\nu}} - \hat{\phi}_{n-\hat{\nu}}) \right]$$

$$\equiv - \sum_{n,m} \hat{\phi}_n^\dagger K_{nm} \hat{\phi}_m - \sum_n \hat{\phi}_n^\dagger \left[-iag_0 A_\nu(n-\hat{\nu}) \hat{\phi}_{n-\hat{\nu}} + iag_0 A_\nu(n) \hat{\phi}_{n+\hat{\nu}} \right]$$

(eqs. (6.3), (6.5))

$$+ \sum_n \hat{\phi}_n^\dagger \hat{\phi}_n m^2 - \sum_n \hat{\phi}_n^\dagger (iag_0 A_\nu)^2 \hat{\phi}_n$$

$$\rightarrow - \int d^4x \phi^\dagger(x) \partial_\nu^2 \phi(x) \quad (6.23)$$

Remarks (1) The definition $\hat{\phi}(n) = iag_0 A_\nu(n)$ is only 'unique' up to order a^2 .

(2) Field strength $F_{\mu\nu}$:

(a) in continuum:

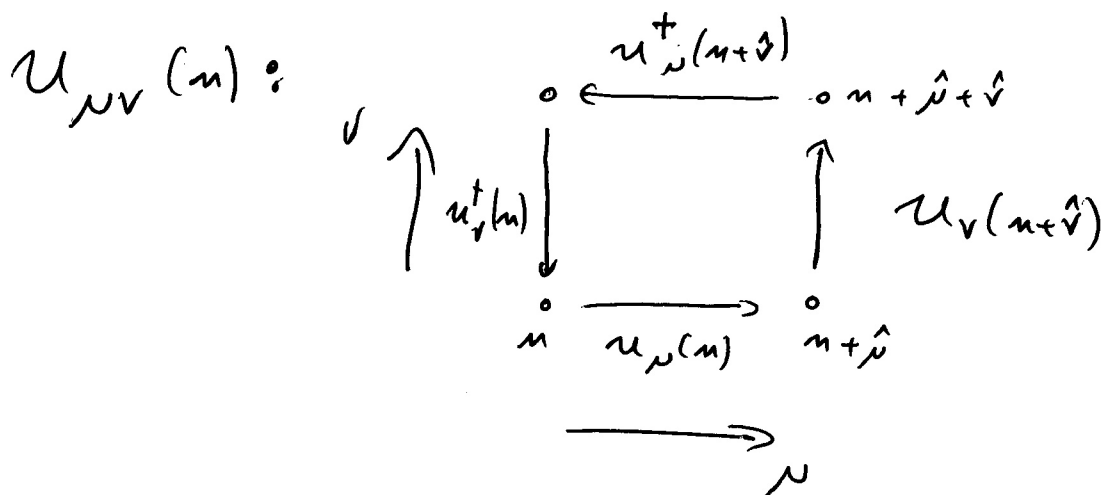
$$\text{curvature } \frac{1}{ig} [\partial_\mu, \partial_\nu]$$

(b) 'Curvature' on the lattice

$$U_{\mu\nu}(n) = U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n)$$

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 U_p Plaquette variable

(6.24)



We define in analogy

$$U_{\mu\nu}(n) = e^{i g_0 a^2 \mathcal{F}_{\mu\nu}(n)} \quad (6.25)$$

with lattice fieldstrength tensor $\mathcal{F}_{\mu\nu}(n)$.

In QED we derive from eq. (6.24),

$$\mathcal{F}_{\mu\nu}(n) = \frac{1}{a} \left[\overbrace{(A_\nu(n + \hat{\nu}) - A_\nu(n))}^{\hat{\partial}_\nu^R A_\nu} - \overbrace{(A_\mu(n + \hat{\nu}) - A_\mu(n))}^{\hat{\partial}_\nu^R A_\mu} \right] \quad (6.26)$$

In QCD we can use Baker-Campbell-Hausdorff

$$e^A e^B = e^{A+B + \frac{1}{2}[A,B] + \dots} \quad (6.27)$$

and arrive at

$$\mathcal{F}_{\mu\nu}(n) = \frac{1}{a} \left[\hat{\partial}_\nu^R A_\mu - \hat{\partial}_\mu^R A_\nu + i g_0 a [A_\mu, A_\nu](n) + \mathcal{O}(a) \right] \quad (6.28)$$

This allows us to derive an lattice analogue of the pure Yang-Mills action,

$$S_W[U_\rho] = \beta \sum_{\mu < \nu} \left(1 - \frac{1}{2N} \text{tr}_{\text{fund}} (U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n)) \right) \quad (6.28)$$

with

$$\beta = 2N/g_0^2 \quad (6.29)$$

This follows straight forwardly from

$$U_{\mu\nu}(n) \xrightarrow{a \rightarrow 0} 1 + i g_0 a \mathcal{F}_{\mu\nu}(n) - \frac{g_0^2 a^2}{2} \mathcal{F}_{\mu\nu}^2(n) + \mathcal{O}(a^3) \quad (6.30)$$

and hence

$$U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n) = -g_0^2 a^2 \underbrace{\mathcal{F}_{\mu\nu}^2(n)}_{\text{no. sum}} + \mathcal{O}(a^3) \quad (6.31)$$

Inserting eq. (6.31) in (6.28) we are led to

$$S_W[U_\rho] \xrightarrow{a \rightarrow 0} S_{YM}[A] = \frac{1}{2} \int d^4x \text{tr}_{\text{fund}} F^2 \quad (6.32)$$

$$\begin{aligned} \sum_{\mu < \nu} \left(1 - \frac{1}{2N} \text{tr} (U_{\mu\nu} + U_{\mu\nu}^\dagger) \right) &= \sum_{\mu < \nu} \frac{g_0^2 a^2}{2N} \text{tr} \mathcal{F}_{\mu\nu}^2 + \mathcal{O}(a^3) \\ &= g_0^2 a^2 / 4N \text{tr} \mathcal{F}_{\mu\nu}^2 \leftarrow \text{sum} + \mathcal{O}(a^3) \end{aligned}$$

Lattice generating functional for YM-theory:

$$Z \approx \int \mathcal{D}U e^{-S_W[U]} \quad (6.33)$$

and

$$\begin{aligned} & \langle U_{\mu_1}^{a_1 b_1}(u_1) \dots U_{\mu_m}^{a_m b_m}(u_m) \rangle \\ &= \frac{1}{Z} \int \mathcal{D}U U_{\mu_1}^{a_1 b_1}(u_1) \dots U_{\mu_m}^{a_m b_m}(u_m) e^{-S_W[U]} \end{aligned} \quad (6.34)$$

where

$$\mathcal{D}U = \prod_l dU_l \quad \begin{array}{l} \leftarrow \text{Haar measure} \\ \uparrow \\ l \leftarrow \text{links} \end{array}$$

with

$$\int dU u^{ab} = 0$$

$$\int dU u^{ab} u^{cd} = 0$$

(6.35)

$SU(3)$:

$$\int dU u^{ab} (u^\dagger)^{cd} = \frac{1}{3} \delta_{ad} \delta_{bc}$$

$$\int dU u^{a_1 b_1} u^{a_2 b_2} u^{a_3 b_3} = \frac{1}{3!} \varepsilon_{a_1 a_2 a_3} \varepsilon_{b_1 b_2 b_3}$$

and

$$\int dU^V = \int dU = 1 \quad (6.36)$$

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 VUV^\dagger

$$\int dU \underbrace{u^{ab} (u^\dagger)^{bd}}_{\delta_{ad}} = \delta_{ad}$$