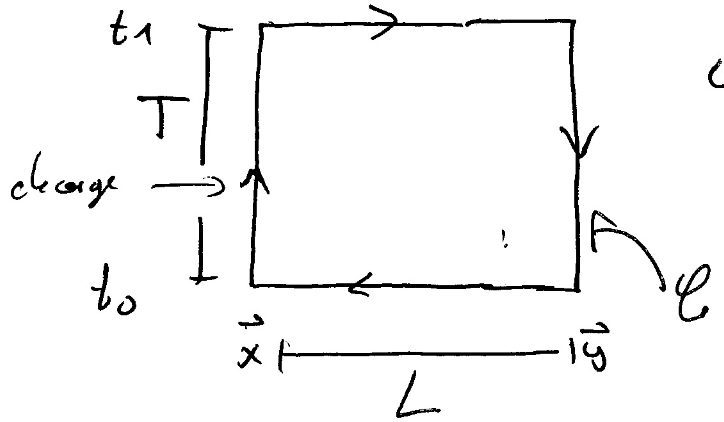
6.3 The Wegner-Wilson loop & the static quark potential

First consider an electron-positron pair, which is created at some initial time, pulled apart, kept at some distance L and then annihilated. This can be described by coupling a current of the world lines of the e^+e^- pair to the photon:



with $\int d^4x \, J_\mu A_\mu$

$$= ie \int_{t_0}^{t_1} \left(A_0(t,\vec{x}) - A_0(t,\vec{y}) \right) dT$$

$$+ ie \int_{\vec{x}}^{\vec{y}} d\vec{z} \left(\vec{A}(t_1,\vec{z}) - \vec{A}(t_0,\vec{z}) \right) \qquad (6.37)$$

It follows that

$$J_\mu(x) \doteq ie \int_{\mathcal{C}} dz_\mu \, \delta^{(4)}(z-x) \qquad (6.38)$$

We conclude that

$$e^{\int d^4x\, J_\nu A_\nu} = \underbrace{e^{ie\int_{\mathcal{C}} dz_\nu A_\nu(z)}}_{\text{Wegner - Wilson Loop } W_{\mathcal{C}}} \qquad (6.39)$$

Note that eq. (6.39) is gauge invariant:

$$W_{\mathcal{C}}(A^{G}) = e^{ie\int_{\mathcal{C}} dz_\nu A^{G}_\nu(z)} = e^{ie\int_{\mathcal{C}} dz_\nu (A_\nu - 1/e\, \partial_\nu \omega)}$$

$$= e^{ie\int_{\mathcal{C}} dz_\nu A_\nu} \qquad (6.40)$$

$G = e^{i\omega}$

Note also that an infinitesimal open Wilson <u>line</u>

$$W_{\mathcal{C}_{x,y}}(A) = e^{ie\int_{\mathcal{C}_{x,y}} dz_\nu A_\nu(z)} \qquad (6.41)$$

relates to a U(1)-link variable, under a gauge transformation $G(x)$, it transforms

$$W_{\mathcal{C}_{x,y}}(A) = G(x)\, W_{\mathcal{C}_{x,y}}(A)\, G^{+}(y) \qquad (6.42)$$

This allows to define gauge-invariant correlation-functions such as

$$\langle \bar{\Psi}(x)\, W_{\mathcal{C}_{x,y}}(A)\, \Psi(y) \rangle \qquad (6.43)$$

Remarks:

(1) In $SU(N)$ the property eq. (6.42) relevant for gauge invariance requires

$$\mathcal{U}_{\mathcal{C}_{x,y}} = \underset{\substack{\uparrow \\ \text{path ordering (remember time-ord.)}}}{\mathcal{P}} e^{ig \int_{\mathcal{C}_{x,y}} dz_\nu A_\nu(z)} \tag{6.44}$$

(2) The covariant derivative follows as

$$\mathcal{U}^\dagger_{\mathcal{C}_{y,x}} \partial_\nu \, \mathcal{U}_{\mathcal{C}_{y,x}} = \partial_\nu + ig A_\nu \tag{6.45}$$

This property can be used to write the Dirac eq. in terms of a phase factor $W_\mathcal{C}$ and the free Dirac eq..

In summary we conclude that the expectation value of a static $q\bar{q}$-pair is given by

$$\boxed{W[L,T] = \frac{1}{Z}\int dA \; \underset{\substack{\uparrow \\ \mathrm{tr}\,\mathcal{U}_\mathcal{C}}}{W_\mathcal{C}(A)} \; e^{-S_{YM}[A]}} \tag{6.46}$$

see p. 141

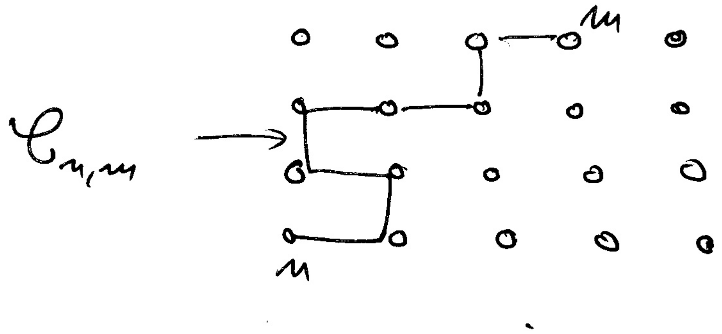
On the lattice this is given by

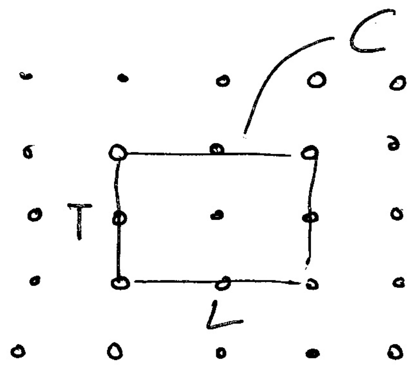
$$W[L,T] = \frac{1}{Z} \int \mathcal{D}\mathcal{U} \cdot W_C[\mathcal{U}]\, e^{-S_W[\mathcal{U}]} \qquad (6.47)$$

where

$$W_C[\mathcal{U}] = \mathrm{tr}\, \mathcal{U}_C$$

$$\text{with} \quad \mathcal{U}_{C_{n,m}} = \prod_{\ell \in C_{n,m}} \mathcal{U}_\ell \qquad (6.48)$$





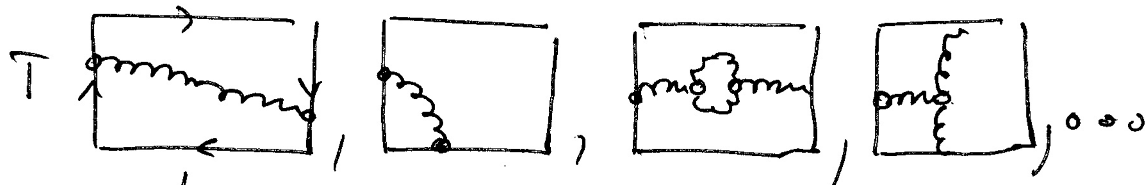
Interpretation:

$$\boxed{\lim_{T\to\infty} W[L,T] = F(L) \cdot e^{-E(L)\,T}} \qquad (6.49)$$

interaction energy

overlap with ground state

Remark: The interaction energy is perturbatively due to diagrams such as



, ...

Static quark potential in the strong coupling expansion

We want to compute (eq. 6.46))

$$W[L,T] = \frac{\int \mathcal{D}\mathcal{U}\, W_C[\mathcal{U}]\, e^{\beta \sum_p S_p}}{\int \mathcal{D}\mathcal{U} \cdot e^{\beta \sum_p S_p}} = \langle W_C[\mathcal{U}]\rangle \quad (6.50)$$

where

$$S_p = \frac{1}{2N} \mathrm{tr}\left(\mathcal{U}_p + \mathcal{U}_p^+\right) \quad (6.51)$$

Plaquettes : $\mathcal{U}_p = \mathcal{U}_{\mu\nu}$ (see eq. 6.24))

in an expansion about $\beta = \frac{2N}{g_0^2} \to 0 \Longleftrightarrow g_0 \to \infty$.

The action factor $e^{\beta \sum_p S_p}$ is expanded as

$$e^{\beta \sum_p S_p} = \prod_p e^{\beta S_p} = \prod_p \sum_n \frac{\beta^n}{n!} S_p^n \quad (6.52)$$

Now we use the integration rules eq. (6.35), (6.36)

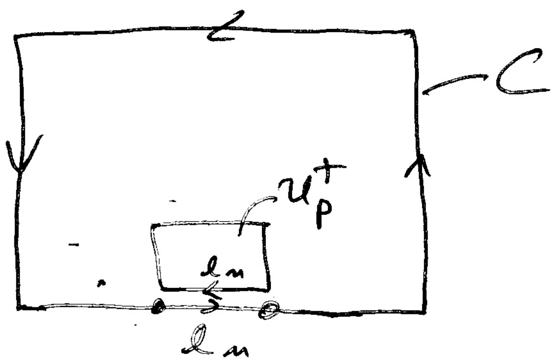
of the Haar measure:

(1) Expanding the denominator of eq. (6.50)

in powers of β leads to

$$\int \mathcal{D}\mathcal{U}\, e^{\beta \sum_p S_p} = \int \mathcal{D}\mathcal{U} + \mathcal{O}(\beta) \quad (6.53)$$

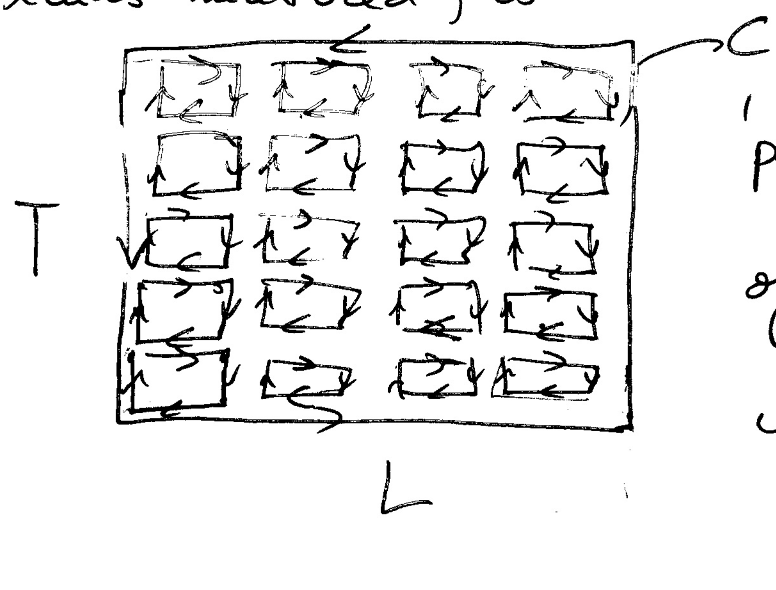
(2) The link variables U_l along the Wegner-Wilson loop have to be matched by U_l^+, otherwise the group integration $\int dU_l \, U_l$ vanishes!



$\rightarrow$ direction of link

Inserting an U_p^+ from the expansion of $e^{\beta \sum_p S_p}$ matches the link variable U_{l_n}, but also creates 3 free links U_l^+ which have to be matched.

Hence, the smallest number of plaquettes with all links matched, is



'paving the inside of the loop C with plaquettes'

The loop is paved with $\underset{\hat{T}}{T/a} \cdot \underset{\hat{L}}{L/a} = \frac{\overset{\text{Area}}{A}}{a^2} = \hat{A}$

plaquettes, and hence we conclude that

$$W[T,L] = \prod_{\ell \in C} \int dU_\ell \, \underbrace{U_\ell^{a_\ell b_\ell}}_{\text{trace over } C} \underbrace{(U_\ell^+)^{c_\ell d_\ell}}_{\text{traces over } P\text{'s}} \cdot \Big(\underbrace{\beta/6}_{2N=6,\ SU(3)}\Big)^{\hat{A}} + \mathcal{O}(\beta^{\hat{A}+1}) \qquad (6.53)$$

The group integrations give factors

$(1/3)^{2\hat{A}+\hat{L}+\hat{T}}$, the traces factors $3^{\overbrace{(\hat{L}+1)(\hat{T}+1)}^{\#\text{ lattice sites}}}$

In summary we arrive at the final result,

$$\boxed{W[T,L] = 3 \cdot \left(\frac{\beta}{18}\right)^{\hat{A}} + \mathcal{O}(\beta^{\hat{A}+1})} \qquad (6.54)$$

The $q\bar{q}$-potential is then given by

$$\boxed{\hat{V}(\hat{L}) = -\lim_{\hat{T}\to\infty} \frac{1}{\hat{T}} \ln \langle W_C[U] \rangle = \hat{\sigma}(g_0)\hat{L}} \qquad (6.55)$$

with string tension $\boxed{\hat{\sigma} = -\ln \beta/18}$ (6.56)

Remarks:

(1) The computation leading to eq. (6.55) also go through for $U(1)$. Accordingly, <u>compact</u> $U(1)$ has a confining phase on the lattice. It also has a Coulomb phase ($1/r$-pot). Indeed, these phases are separated by a phase transition. In QCD we do not want to have this!

(2) The above already raises the question about the continuum limit. As discussed on p. 134, we have to tune the lattice coupling $g_0(a)$ for approaching the cont. lim.. We will see that

$$g_0 \xrightarrow{\text{cont. lim.}} 0 \qquad (6.57)$$