

6.5 Fermions on the lattice

We conclude our Lattice excursion with a few remarks on fermions. We put fermions on the lattice similarly to scalars:

$$S_D[\psi, \bar{\psi}] = - \int d^4x \bar{\psi}(\not{\partial} + m) \psi \quad (6.80)$$

see eq. (2.30), p. 49. The lattice substitutions are analogously to eq. (6.3) for scalars:

$$\begin{aligned} \bar{\psi}(x) &\rightarrow \frac{1}{a^{3/2}} \hat{\bar{\psi}}(n) \\ \partial_\rho \psi(x) &\rightarrow \frac{1}{a^{5/2}} \hat{\partial}_\rho \hat{\psi}_n \\ n &\rightarrow \frac{1}{a} \hat{n} \end{aligned} \quad (6.81)$$

with

$$\hat{\partial}_\rho \hat{\psi}(n) = \frac{1}{2} (\hat{\psi}_{n+\hat{\rho}} - \hat{\psi}_{n-\hat{\rho}})$$

being the symmetric lattice derivative, see eq. (6.5).

Then, the lattice action reads

$$S_D = \sum_{\substack{m, n \\ \alpha, \beta \\ \uparrow \uparrow \\ \text{spinor indices}}} \overline{\hat{\psi}}_{\alpha, m} K_{\alpha\beta, mn} \hat{\psi}_{\beta, n} \quad (6.82)$$

with

$$K_{\alpha\beta, mn} = - \sum_{\nu} \frac{1}{2} (\gamma_{\nu})_{\alpha\beta} [\delta_{m, n+\hat{\nu}} - \delta_{m, n-\hat{\nu}}] + \hat{m} \delta_{mn} \delta_{\alpha\beta} \quad (6.83)$$

and the generating functional is given by

$$Z[\eta, \bar{\eta}] \approx \int \prod_{m, \alpha} \overline{\eta}_{m, \alpha} d\hat{\psi}_{\alpha, m} \prod_{m, \beta} d\overline{\hat{\psi}}_{\beta, m} e^{-S_D + \sum_{m, \alpha} (\bar{\eta}_{m, \alpha} \hat{\psi}_{\alpha, m} - \overline{\hat{\psi}}_{\alpha, m} \eta_{m, \alpha})} \quad (6.84)$$

The Grassmann integrals in eq. (6.84) are performed

straight forwardly and yield

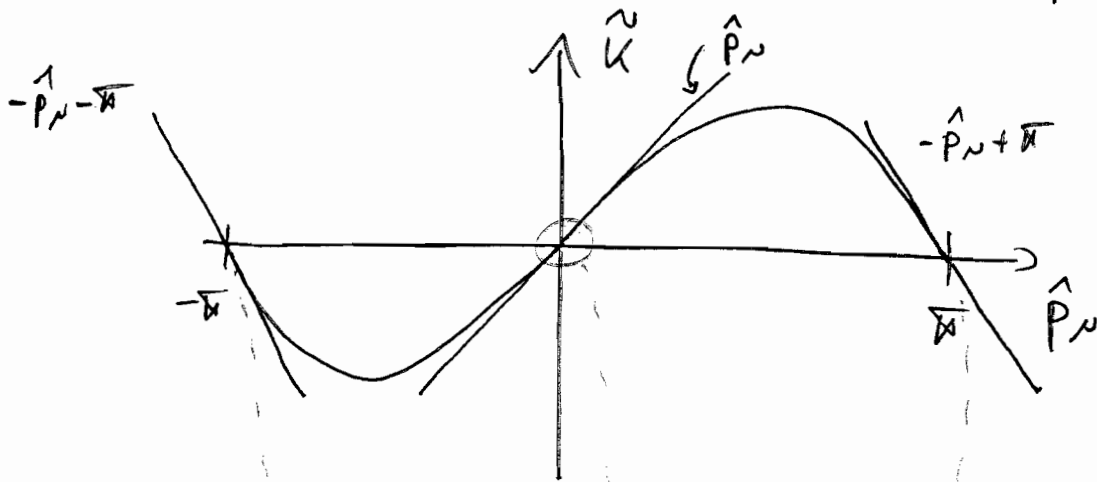
$$Z[\eta, \bar{\eta}] \approx \det K \cdot e^{-\sum_{m, n, \alpha, \beta} \bar{\eta}_{\alpha, m} K_{\alpha\beta, mn}^{-1} \eta_{\beta, n}} \quad (6.85)$$

It is left to invert K . As in the scalar case we go to momentum space:

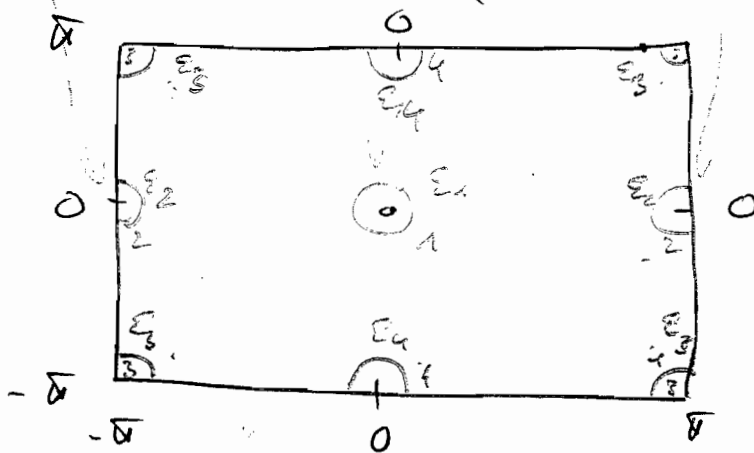
$$K_{\alpha\beta, mn} = \int_{-\pi}^{\pi} \frac{d^4 \hat{p}}{(2\pi)^4} \tilde{K}_{\alpha\beta}(\hat{p}) e^{i\hat{p}(n-m)} \quad (6.86)$$

with $\tilde{K}_{\alpha\beta}(\hat{p}) = -\sum_{\nu} (\gamma_{\nu})_{\alpha\beta} \sin \hat{p}_{\nu} - \hat{m}$ (6.87)

⇒ Dispersion relation: (all momenta $\hat{p}_{\nu} = 0$ for $\nu \neq \mu$)



Zeros in two dim: (Brillouin zone)



⇒ 4 fermions

in d dim

2^d fermions

E_i : neighbourhood of zero

Continuum limit for propagator: (see eq. (6.14))

$$\begin{aligned}
 & \lim_{a \rightarrow 0} \frac{1}{a^3} \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} \frac{e^{i \hat{p} \cdot (n-m)}}{\tilde{K}(\hat{p})} \\
 &= - \lim_{a \rightarrow 0} \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} \frac{e^{i p \cdot (x-y)}}{\gamma_\nu \frac{1}{a} \sin a p_\nu + m} \quad (6.88) \\
 &= - \sum_i \int_{\Sigma_i} \frac{d^4 p}{(2\pi)^4} \frac{e^{i p \cdot (x-y)}}{(-1)^{i+1} \gamma_\nu p_\nu + m} + \mathcal{O}(a)
 \end{aligned}$$

Note that the sign of the γ -matrices can be flipped by a \hat{p} -dependent similarity transformation:

$$\Upsilon_{\hat{p}} \gamma_\nu \Upsilon_{\hat{p}}^{-1} = (-1)^{i+1} \gamma_\nu \quad (6.89)$$

In summary we have produced 16 fermions on the lattice. This is the infamous doubling problem on the lattice.

Remarks:

(1) For massive fermions we can suppress the 15 doublers with ($\hat{\Delta}$ in (6.4), (6.7))

$$\frac{1}{2} \sum_{\substack{\alpha, \beta \\ \alpha, \beta}} \bar{\Psi}_{\alpha, m} \hat{\Delta}_{\alpha, m} \Psi_{\beta, m} \quad (6.90)$$

Wilson term $\hat{\Delta} = \sum_{\mu > 0} [\partial_{\mu+\hat{\mu}, m} + \partial_{\mu-\hat{\mu}, m} - 2\partial_{\mu, m}]$

It diverges like $\frac{1}{a} \epsilon_i, i \neq 1, \dots, d$

vanishes with a in ϵ_1 . [No chiral sym.]

(2) This does not work for chiral fermions: Nielsen-Ninomiya no-go theorem

'One cannot put one Weyl fermion on the lattice'

Ways out:

(a) break chiral sym. a la (6.90) on the lattice; numerical costs high for recovering chiral sym. in cont. limit

(b) maximal chiral invariance: $S_D = \sum_{\substack{\mu, \nu \\ \alpha, \beta}} \bar{\psi}_{\alpha, \mu} \mathcal{D}_{\beta, \nu \mu} \psi_{\beta, \nu}$
 Ginsparg-Wilson relation (RG-symmetry)

Consider Dirac operators \mathcal{D} on the lattice
 with

$$\gamma_5 \mathcal{D} + \mathcal{D} \gamma_5 = 2a \mathcal{D} \gamma_5 \mathcal{D} \quad (6.91)$$

Eq. (6.91) allows the definition of -deformed-
 chiral transformations

$$\psi \rightarrow e^{i\alpha \tilde{\gamma}_5} \psi \quad (6.92)$$

with $\tilde{\gamma}_5 = (1 - a/\mathcal{D}) \gamma_5$. Exercise

Numerical costs: solutions of eq. (6.91) involve
 roots \rightarrow high numerical costs

(c) staggered fermions (a standard choice)

look up books for details

Numerical costs: relatively low

problems: relation to physics theory

(a) 'taste violation'

(b) anomaly counting

(3) The Dirac operator having positive and negative eigen values causes a numerical problem (in particular in the chiral limit),

$$\det(D + m) \geq 0$$

the so called sign problem.

For non-vanishing density (or chemical pot. μ)

$$D \rightarrow D + i\gamma_0 \mu$$

this is most severe: NP-hard

↑
Non-deterministic Polynomial time
-hard