7 Renormalisation group

We have seen in the last chapters, in particular in 6.4, how to use relations, that governed the dependence of correlation functions & observables on physics scales, such as the distance L of a $q\bar{q}$-pair, as well as the independence of physics on regularisation scales such as the lattice spacing a.

On the lattice the latter is formalised in the Kadanoff block spin transformations

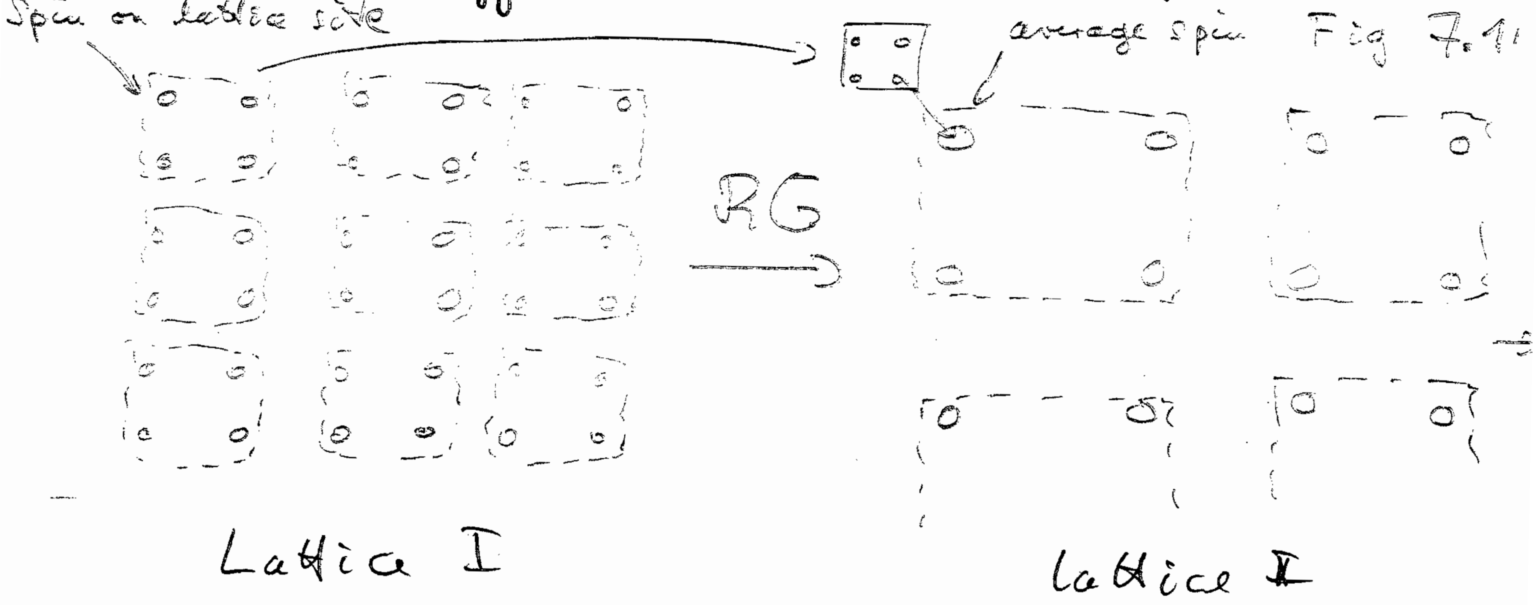


Fig 7.1:

Also rescaling the lattice parameters (couplings) & fields leads to the same theory, keeping the scales halves (or doubles) the coarse graining scale a. Iterating this procedure, the system 'runs' into a fixed point of the renormalisation group map RG.

7.1 Wilson's renormalisation group

Now we transport this picture to the continuum, with a momentum cut-off $\Lambda \sim 1/a$, and a blocking step going from $\Lambda \to b\Lambda$. On the level of the gen. funct. (b = ½ in Fig 7.1) this means (without source terms)

$$Z_\Lambda = \int [\mathcal{D}\phi]_\Lambda \, e^{-\overbrace{\int d^dx \left\{ \frac{1}{2}\phi(-\Delta+m^2)\phi + \lambda/4!\,\phi^4 \right\}}^{S[\phi]}} \qquad (7.1)$$

with $[\mathcal{D}\phi]_\Lambda = \prod_{p^2 \leq \Lambda} d\phi(p)$

or $\phi(p^2 \geq \Lambda^2) = 0$

Now we perform the RG step. To that end we define the field $\hat{\phi}(p)$ with

$$\hat{\phi}(p) = \begin{cases} \phi(p) & b\Lambda \leq |p| \leq \Lambda \\ 0 & \text{else} \end{cases} \qquad (7.2)$$

Now we write the funct. integral measure in eq. (7.1) as

$$[\mathcal{D}\phi]_\Lambda = [\mathcal{D}\phi]_{b\Lambda}\, \mathcal{D}\hat{\phi} \qquad (7.3)$$

and hence

$$Z_\Lambda = \int [\mathcal{D}\phi]_{b\Lambda} \int \mathcal{D}\hat{\phi}\, e^{-\int d^dx \left\{ \frac{1}{2}(\phi+\hat{\phi})(-\Delta+m^2)(\phi+\hat{\phi}) + \lambda/4!\,(\phi+\hat{\phi})^4 \right\}}$$

$$= \int [\mathcal{D}\phi]_{b\Lambda}\, e^{-S[\phi]}$$

$$\cdot \int \mathcal{D}\hat{\phi}\, e^{-\int d^dx \left\{ \frac{1}{2}\hat{\phi}(-\Delta+m^2)\hat{\phi} + \lambda\left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right) \right\}}$$

$$(7.4)$$

where we have used that (prove with Four. trafo.)

$$\int d^dx\, \hat{\phi}\,\Delta\phi = \int d^dx\, \hat{\phi}\,\phi\, m^2 = 0$$

Integrating formally over $\hat{\phi}$ leads to

$$Z_\Lambda = \int [\mathcal{D}\phi]_{b\Lambda}\, e^{-S_{eff}[\phi]} \qquad (7.5)$$

with $e^{-S_{eff}[\phi]} = e^{-S[\phi]} \cdot \int \mathcal{D}\hat{\phi}\, e^{-\int d^dx \{\frac{1}{2}\hat{\phi}(-\Delta+m^2)\hat{\phi}+\dots\}}$

Now we assume $\Lambda, b\Lambda$ being bigger than the physics scales, e.g. $m^2/\Lambda^2 \ll 1$. Then we can expand the $\hat{\phi}$-integral in eq. (7.5) about

$$\int d^dx\ \hat{\phi}\,(-\Delta)\,\hat{\phi} = \int_{b\Lambda}^{\Lambda} \frac{d^dp}{(2\pi)^4}\, \hat{\phi}(p)\, p^2\, \hat{\phi}(-p) \qquad (7.6)$$

$\hookrightarrow b^2\Lambda^2 \leq p^2 \leq \Lambda^2$

An approximate saddle point is $\hat{\phi} \sim \frac{1}{\Lambda} \to 0$ and hence (see chapter 3*)

$$\int \mathcal{D}\hat{\phi}\, e^{-\int d^dx \{\frac{1}{2}\hat{\phi}(-\Delta+m^2+\frac{1}{2}\phi^2)\hat{\phi} + \underbrace{\frac{\lambda}{6}\phi^3}_{J(\phi)}\hat{\phi} + O(\hat{\phi}^3)\}} \qquad (7.7)$$

$$\simeq {\det}'^{-1/2}\left(-\Delta+m^2+\frac{\lambda}{2}\phi^2\right) e^{O(1/\Lambda)} \leftarrow \frac{1}{2}\int_{x,y} \frac{\lambda}{6}\phi^3 \frac{1}{-\Delta+m^2+\frac{\lambda}{2}\phi^2} \frac{\lambda}{6}\phi^3$$

$J \longrightarrow \frac{\lambda}{6}\phi^3$

$\uparrow$ runs over spectral values $b^2\Lambda^2 \leq p^2 \leq \Lambda^2$

The det in eq. (7.7) can be written in terms of a trace ($\det A = e^{\operatorname{Tr} \ln A}$), to wit

$$\det{}^{-1/2}\left(-\Delta + m^2 + \lambda/2\,\phi^2\right) = e^{-1/2 \operatorname{Tr}' \ln\left(-\Delta + m^2 + \lambda/2\,\phi^2\right)}$$

$\uparrow$ runs over $b^2\Lambda^2 \leq p^2 \leq \Lambda^2$

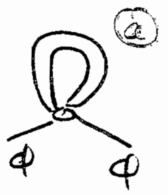
(7.8)

and hence

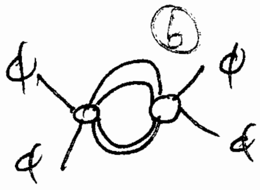
$$S_{eff}[\phi] = S[\phi] + \frac{1}{2} \operatorname{Tr}' \ln\left(-\Delta + m^2 + \lambda/2\,\phi^2\right) + O(1/\Lambda) \qquad (7.9)$$

Diagrammatically this yields

$$S_{eff}[\phi] = S[\phi] +$$



$$+$$



$$+ O(\phi^6)$$

(7.10)

with $$= \; = \left.\frac{1}{p^2 + m^2}\right|_{p^2 \in [b^2\Lambda^2, \Lambda^2]} \simeq \frac{1}{p^2}\left(1 + O\left(m^2/\Lambda^2\right)\right)$$

The first two terms are, ⓐ $= \frac{1}{2}\mu^2 \int d^d x\, \phi^2$ with

$$\mu^2 \simeq \frac{\lambda}{4} \int_{b\Lambda}^{\Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} = \frac{\lambda}{(4\pi)^{d/2}\,\Gamma(d/2)} \frac{1 - b^{d-2}}{d-2} \Lambda^{d-2}$$

(7.11)

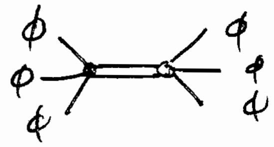
and $+ 1/4! \, h \int d^dx \, \phi^4$

$$h \simeq -4! \frac{1}{2} \frac{1}{2} \left(\frac{\lambda}{2}\right)^2 \int \frac{d^dp}{(2\pi)^d} \left(\frac{1}{p^2}\right)^2$$

$$= -\frac{3\lambda^2}{(4\pi)^{d/2}\Gamma(d/2)} \cdot \frac{(1-b^{d-4})}{d-4} \Lambda^{d-4} \qquad (7.12)$$

$$\xrightarrow[d\to 4]{} -\frac{3\lambda^2}{16\pi^2} \ln 1/b$$

From the 'source' term in eq. (7.7) we also get terms with ϕ^6 (and higher),



$$\sim \frac{1}{\Lambda^2} \int_x \phi^6 \qquad (7.13)$$

There are also derivative terms such as

$$\eta \int_x \phi^2 (\partial_\nu \phi)^2 \qquad (7.14)$$

with $\eta \sim 1/\Lambda^2$. In summary we have

$$S_{eff}[\phi] = S_{cl}[\phi] + \text{connected diagrams} \qquad (7.15)$$

In the diagrams (coefficients) ⓐ & ⓑ computed in eqs. (7.11), (7.12) we see how the coefficient of the classical action in the generating functional change, if a momentum shell $p^2/\Lambda^2 \in [b^2, 1]$ is integrated out: the UV-cut-off is lowered by the factor b.

From eq. (7.14) we deduce that

$$S_{eff}[\phi] = \int d^d x \left\{ \frac{1}{2} (1+\overset{\text{only at 2-loop, or } 2^{nd} \text{ it.}}{\Delta Z}) \phi(-\Delta)\phi + \frac{1}{2}\left(m^2 + \underbrace{\Delta m^2}_{\mu^2 \leftarrow \text{eq.}(7.11)}\right)\phi^2 \right.$$

$$\left. + \frac{1}{4!}(1+\underbrace{\Delta\lambda}_{+h \in 1\text{-loop, eq. }(7.12)})\phi^4 + \overset{\alpha=0}{(\alpha+\Delta\alpha)}(\partial_\nu\phi)^4 + \overset{\lambda_6=0}{\frac{\lambda_6+\Delta\lambda_6}{6!}}\phi^6 + \dots \right. \tag{7.16}$$

Now we want to exploit and maximise the similarity of the generating functional as represented in eq. (7.1) and eq. (7.5).

To that end we <u>re-parameterise</u> our theory such that

$$S_{\text{eff}}[\phi] = S[\phi'] + \text{higher-order terms}$$

where $\phi(p^2 > b^2\Lambda^2) = 0$ and $\phi'(p'^2 > \Lambda^2) > 0$.

This is achieved by the rescalings

$$p' = p/b \;, \quad x' = x b$$

(see lattice on p. 164)

(7.17)

and

$$\phi'(x') = \left[b^{2-d}(1+\Delta Z)\right]^{1/2} \phi(x)$$

$$m'^2 = (m^2 + \Delta m^2)\,\frac{1}{1+\Delta Z}\; b^{\boxed{-2}}$$

$$\lambda' = (\lambda + \Delta\lambda)\,\frac{1}{(1+\Delta Z)^2}\; b^{\boxed{d-4}}$$

$$\alpha' = (\alpha + \Delta\alpha)\,\frac{1}{(1+\Delta Z)^2}\; b^{\boxed{d}}$$

$$\lambda_6' = (\lambda_6 + \Delta\lambda_6)\,\frac{1}{(1+\Delta Z)^3}\; b^{\boxed{2d-6}} \qquad (7.18)$$

momentum dim

4-d

-d

6-2d

ϕ^2

ϕ^4

∂^4

ϕ^6

with

$$\int d^d x' = b^d \int d^d x \;, \quad \frac{\partial}{\partial x'} = \frac{1}{b}\frac{\partial}{\partial x} \qquad (7.19)$$

With the reparameterisations (7.17), (7.18), (7.19) we have

$$S_{eff}[\phi'] = \overbrace{\int d^dx' \left\{ \frac{1}{2} \phi'(x') (-\Delta' + m'^2) \phi' + \frac{\lambda'}{4!} \phi'^4 \right\}}^{S[\phi'; m', \lambda']}$$

$$+ \int d^dx' \left\{ \alpha' (\partial'_\nu \phi')^4 + \frac{\lambda_6'}{6!} \phi'^6 + \dots \right\} \tag{7.20}$$

The generating functional now reads

$$Z_\Lambda = \int [\mathcal{D}\phi']_\Lambda \, e^{-S_{eff}[\phi']} \tag{7.21}$$

and hence has the same form as eq. (7.4) with $S \to S_{eff}$. The (Wilsonian) effective action S_{eff} now encodes also the quantum effects of the momentum shell $[b^2\Lambda^2, \Lambda^2]$. This is not only reflected in the additional terms in the second line in eq. (7.20) but also in the change in the parameters of the theory m, λ, $[\phi]$!

Remark: Assume now that we iterate the above procedure by starting with eq. (7.21) instead of eq. (7.1): The saddle point expansion used in eq. (7.7) still holds (the suppression argument with $1/\Lambda$) and we arrive finally again at $\boxed{\text{eq.}(7.21)!}$ with a Wilson action

$$S_{eff,2} \qquad (7.22)$$

where 2 stands for second iteration.

Repeating this we finally have

$$Z = \int [\mathcal{D}\Phi]_\Lambda \, e^{-S_{eff_\Lambda}[\Phi]} \qquad (7.23)$$

where $S_{eff_\Lambda} = S_{eff,\infty}$, the fixed point of the RG-map with $\Lambda \to b\Lambda$.

What we have done is to *integrate-out* momenta in the shell $[b^2\Lambda^2, \Lambda^2]$ and then we have rescaled the theory such that $b\Lambda \to \Lambda$. Hence we have gone back to the original theory. Then, however, S_{eff_Λ} is **not** the classical action of the theory, but has to encode the integrated-out quantum fluctuations of all momenta above Λ:

(1) intergrate out $p^2 \in [b^2\Lambda^2, \Lambda^2]$

(2) $p \to p/b$: (1) then entails that we have integrated out $p'^2 \in [\Lambda^2, \frac{1}{b^2}\Lambda^2]$

(3) iterate : $(1)^\infty$ then entails integrating-out $p'^2 \in [\Lambda^2, \infty)$

$$\Rightarrow \boxed{\begin{gathered} S_{eff_\Lambda}[\phi] \simeq -\ln \int [D\hat\phi]_{p^2 \geq \Lambda^2}\, e^{-S[\hat\phi + \phi]} \\ \text{with } \phi(p^2 > \Lambda^2) = 0 \qquad (7.24) \end{gathered}}$$

What we have gained is that the non-trivial integration in eq. (7.24) can be done in continuous infinitesimal steps in b and we can monitor the flow of couplings, correlation fcts. & S_{eff_Λ}.

In particular we have

$$\boxed{b \frac{d}{db} S_{eff_\Lambda} = 0,} \qquad (7.25)$$

the renormalisation group equation for S_{eff_Λ}, compare with eq. (6.62) on page 151 on the lattice.

Due to the above arguments, the information of (7.25) also tells us about

$$\boxed{\Lambda \frac{d}{d\Lambda} S_{eff_\Lambda},} \qquad (7.26)$$

the physics change when lowering the cut-off.

As one can see from eq. (7.18), the coefficient $\lambda_{n,m}$ of a term (operator) in the effective action $S_{\text{eff},\Lambda}$ with n fields, ϕ^n, and m derivatives, ∂^m, scales like

$$\lambda_{n,m} \xrightarrow{b} b^{\overbrace{n(d/2-1)+m}^{d_{n,m}}-d}\,\lambda_{n,m} \qquad (7.27)$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$[\phi]\cdot n \qquad [\partial]\cdot m \qquad [d^dx]$$

$p \to p/b$

$$[\lambda] = d - d_{n,m} \longleftarrow [\lambda_{n,m}] + n[\phi] + [\partial]\,m + d = 0$$

An operator with

$$d_{n,m} = n(d/2-1) + m \qquad (7.28)$$

(a) $d - d_{n,m} > 0$, is called relevant.

(b) $d - d_{n,m} = 0$, is called marginal

(c) $d - d_{n,m} < 0$, is called irrelevant

Relevant ops. grow in the iteration (for cut-off $\to \infty$), irrelevant decay, and marginal ones run logarithmically, see eq. (7.12), p. 169.

Renormalisability: $\lambda_{n,m} \sim \Lambda^{d-d_{n,m}}$

(1) Irrelevant couplings decay like $\Lambda^{d-d_{n,m}}$ for $\Lambda \to \infty$.

(2) Relevant and marginal couplings grow with $\Lambda^{d-d_{n,m}}, \ln\Lambda$ for $\Lambda \to \infty$. If the relevant & marginal terms are contained in the classical action, the corrections can be absorbed in redefinitions of the finite number of classical couplings.

We have

$$\boxed{\lim_{\Lambda\to\infty} S_{\mathrm{eff}/\Lambda} \to S_{(\mathrm{bare})}[\phi]} \qquad (7.29)$$

Remark: For a full proof it has to be shown that the notions (a), (b), (c) (rel / marg., irrel.) persist at any it. order.
(order pert. theory)