8 Symmetry breaking in QFT

Symmetry breaking can occur in a QFT in three distinct ways:

(1) Explicit symmetry breaking on the classical level

(2) Spontaneous symmetry breaking: the classical theory has a discrete or continuous symmetry that is broken via the ground state of the theory

(3) Anomalous symmetry breaking: classical symmetries are broken due to topological obstructions on quantum level. These anomalies can be computed - in most cases - perturbatively. However, they are genuinely non-perturbative.

8.1 Spontaneous symmetry breaking

Consider a ϕ^4-theory with N scalar fields and the action

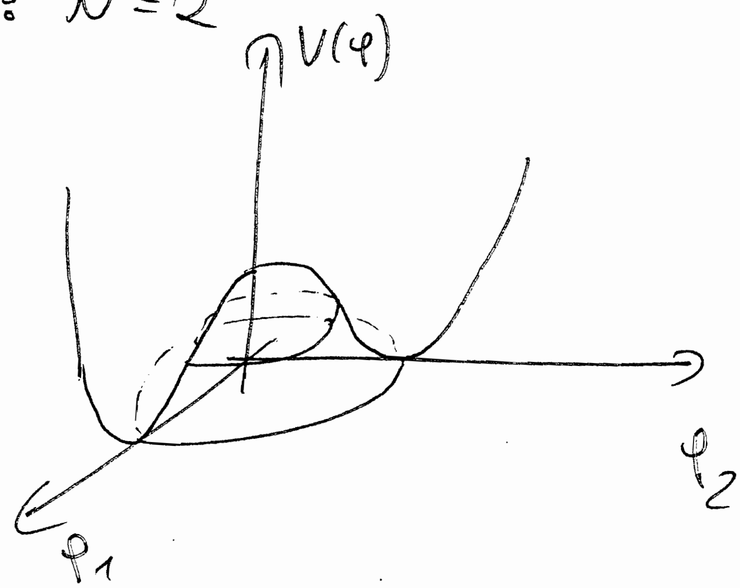
$$S[\varphi] = \int d^dx \left\{ \frac{1}{2} \varphi_i(x)(-\Delta + m^2)\varphi_i(x) + \underbrace{\frac{1}{4}(\varphi_i\varphi_i)^2}_{V(\varphi)} \right\}, \qquad (8.1)$$

the $O(N)$-model, see also chapter 1.4, p. 33-35

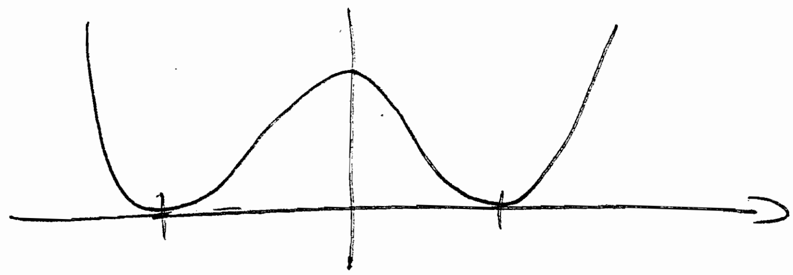
This model has an $O(N)$-symmetry under

$$\varphi_i(x) \to R_{ij}\,\varphi_j(x) \quad \text{with } R \in O(N) \qquad (8.2)$$

Example: $N=2$



Special case $N=1$ (Discrete Z_2 symmetry)



Now we expand our action about the minimum

$$\varphi_0 = (\overbrace{0, \dots, 0}^{N-1}, v) \qquad (8.3)$$

with $v^2 = -m^2/\lambda$ for $m^2 < 0$

Note that also $\varphi_0^R = R \cdot \varphi_0$ is a minimum

with $$\left.\frac{\delta S}{\delta \varphi}\right|_{\varphi_0^R} = 0 \qquad (8.4)$$

(N=4: pions, sigma)

Using $$\varphi(x) = \Big(\underbrace{\vec{\pi}(x)}_{= (\pi_1, \dots, \pi_{N-1})}, v + \sigma(x)\Big) \qquad (8.5)$$

we arrive at

$$S[\vec{\pi}, \sigma] = \int d^dx \Big\{ \frac{1}{2}\sigma(-\Delta \overbrace{-2m^2}^{>0})\sigma + \frac{1}{2}\vec{\pi}(-\Delta)\vec{\pi} + \frac{1}{4}\left[(\vec{\pi}^2)^2 + \sigma^4\right] + v\sigma\vec{\pi}^2 + v\sigma^3 + \frac{1}{2}\vec{\pi}^2\sigma^2 \Big\} \qquad (8.6)$$

mass-less

The original $O(N)$-symmetry is hidden, but still eq. (8.6) shows an $O(N-1)$ symmetry

$$\pi_i \to R_{ij}\,\pi_j\,, \qquad R \in O(N-1) \tag{8.7}$$

for the <u>massless</u> fields π_i with $i=1,\dots,N-1$.

This leads us to Goldstone's Theorem:

'For every spontaneously broken continuous symmetry the theory must contain a massless particle, the Goldstone boson'

Given an action

$$S[\varphi] = S_0[\partial\varphi] + V(\varphi)\cdot \mathrm{Vol}_{\mathbb{R}^d} \tag{8.8}$$

with continuous symmetry: $\boxed{\varphi \to \varphi + \varepsilon\,\Omega(\varphi)}$ (8.9)

$O(N):\ \Omega_i(\varphi) = \omega_{ij}\varphi_j$

$R = e^{\varepsilon\omega}$

and minima φ_0^R with

$$\left.\frac{\partial V}{\partial \varphi}\right|_{\varphi_0} = 0 \tag{8.10}$$

Then we have with

$$\boxed{\Omega_j(\varphi)\frac{\partial V}{\partial \varphi_j} = 0} \tag{8.11}$$

as well as, $\frac{\partial}{\partial\varphi}\left(\Omega\cdot\frac{\partial V}{\partial\varphi}\right)\Big|_{\varphi=\varphi_0}=0$,

$$\frac{\partial \Omega_j}{\partial \varphi_i}\Big|_{\varphi_0}\cdot \underbrace{\frac{\partial V}{\partial \varphi_j}\Big|_{\varphi_0}}_{0} + \Omega_j\Big|_{\varphi_0}\frac{\partial^2 V}{\partial\varphi_i\partial\varphi_j}\Big|_{\varphi_0} = 0 \tag{8.12}$$

or

$$\boxed{\Omega_j\Big|_{\varphi_0}\frac{\partial^2 V}{\partial\varphi_i\partial\varphi_j}\Big|_{\varphi_0} = 0} \tag{8.13}$$

(1) If $\Omega_j(\varphi_0) = 0$, no information can be gained by eq. (8.13) and the ground state φ_0 respects the symmetry

(2) If $\Omega_j(\varphi_0) \neq 0$, the matrix $\frac{\partial^2 V}{\partial\varphi_i\partial\varphi_j}\Big|_{\varphi_0}$ has as many vanishing eigenvalues as the dimension of span $\Omega(\varphi_0)$:

number of Goldstones = number of broken sym.

The quantum case:

On the classical level we have discussed the classical action, its EoM and its symmetries. The quantum EoM's are given by the effective action (see chapter 3), the Legendre transform of $\ln Z[J]$, the free energy,

$$\Gamma[\phi] = \int d^dx \; J\cdot\phi - \ln Z[J]$$

with $\quad \phi = \frac{\delta \ln Z}{\delta J} = \langle\varphi\rangle \qquad (8.14)$

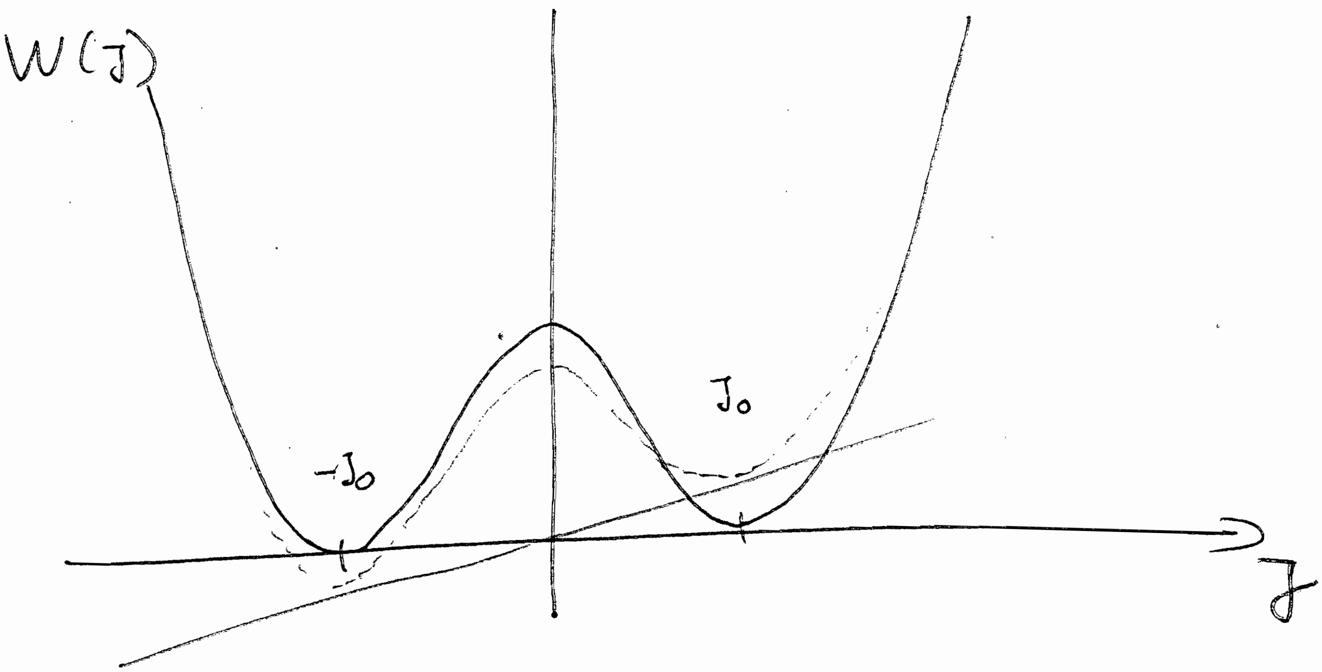
and $\quad J = \frac{\delta\Gamma}{\delta\phi}$, see eqs. (3.2), (3.3), (3.4).

The QEoM is given by

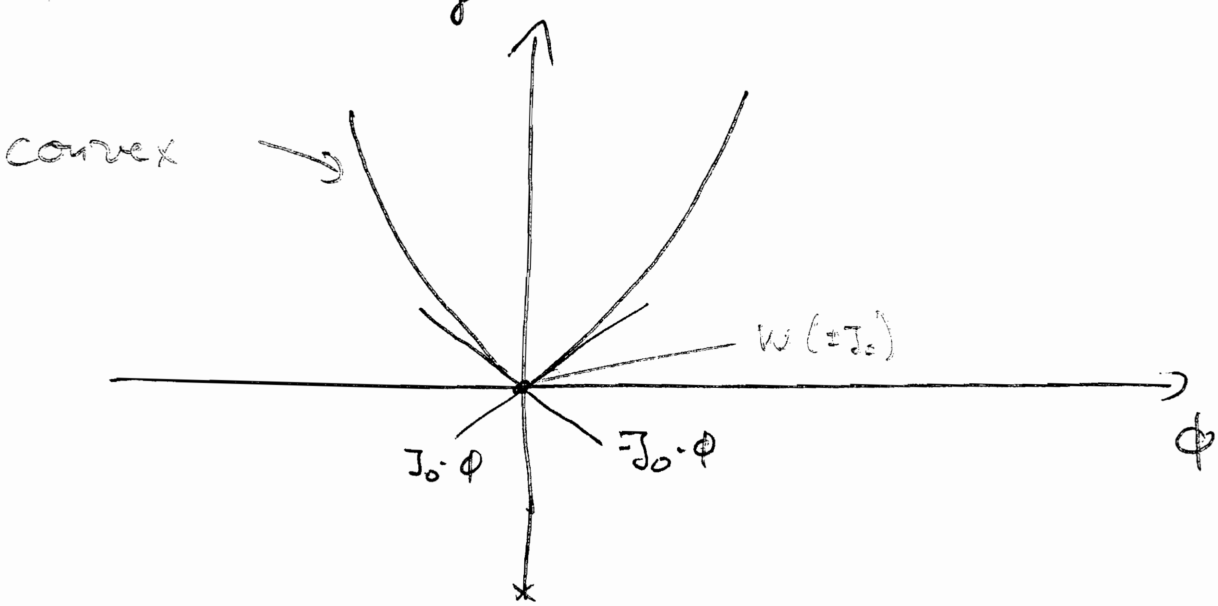
$$\boxed{\left.\frac{\delta\Gamma}{\delta\phi}\right|_{\phi_0} = 0} \qquad (8.15)$$

We expand Γ as in eq. (8.8) and straight away arrive at the Goldstone theorem,

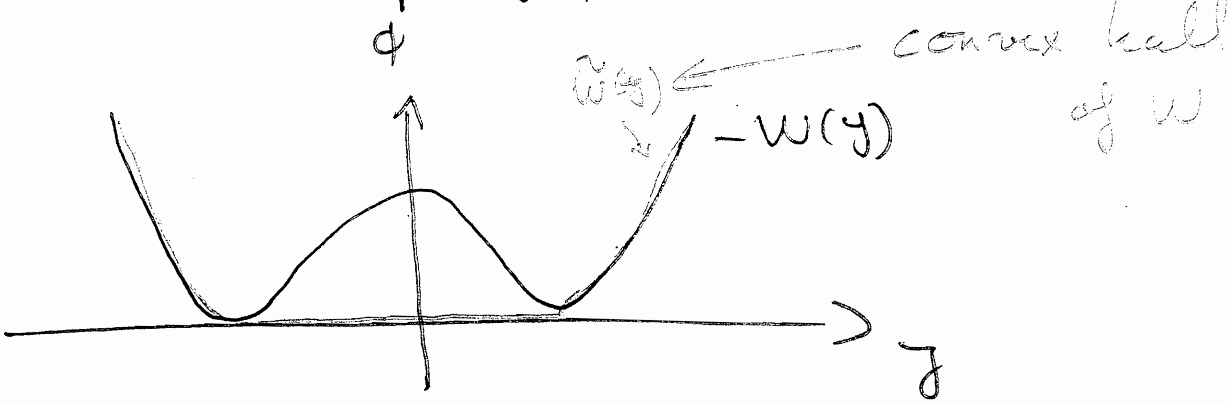
$$\Gamma[\phi] = \Gamma_0[\partial\phi] + V_{eff}[\phi] \qquad (8.16)$$



$$V(\phi) = \sup_J \left(\phi \cdot J - W(J) \right)$$



$$\tilde{W}(J) = \sup_\phi \left(J \cdot \phi - V(\phi) \right)$$



One-loop computation of V_{eff}:

We proceed as in the simple $O(1)$ example in chapter 1.4:

$$\Gamma_{\text{1-loop}}[\phi] = S[\phi] + \frac{1}{2} \operatorname{Tr} \ln \frac{\delta^2 S}{\delta \phi^2} + \text{counter-terms} \tag{8.17}$$

As in chapter 1.4 we can renormalise by taking μ-derivatives. Indeed, now we know that this gives rise to a Callan-Symanzik RG-flow of the effective action Γ.

Recall eq. (1.108) which extends to the present $O(N)$-case as (beware of factors $\frac{1}{4!} \to \frac{1}{4}$)

Exercise

Goldstone

$$\begin{aligned} V_{eff} = {} & \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4} \phi^4 \\ & + \frac{1}{64\pi^2} \Big[(N-1) (\lambda \phi^2 + m^2)^2 \Big(\ln \frac{(\lambda \phi^2 + m^2)}{\mu^2} - \frac{3}{2} \Big) \\ & + (3\lambda \phi^2 + m^2)^2 \Big(\ln \frac{(3\lambda \phi^2 + m^2)}{\mu^2} - \frac{3}{2} \Big) \Big] \end{aligned} \tag{8.18}$$

For $m^2 \to 0$ this simplifies to

$$V_{eff}(\phi) = \lambda/4 \; \phi^4 + \frac{\lambda^2}{6\pi^2}(N+8)\,\phi^4\left(\left[\ln \lambda\phi^2/\mu^2 - \tfrac{3}{2}\right] + 9\ln 3\right) \tag{8.19}$$

The CS-equation now tells us that

$$\boxed{\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - \gamma\phi\frac{\partial}{\partial\phi}\right)V_{eff}(\phi) = 0} \tag{8.20}$$

In 4d we have

dimensionless

$$V_{eff}(\phi) = \phi^4 \cdot u(\phi/\mu, \lambda) \tag{8.21}$$

$[V_{eff}] = 4$ $\qquad$ $[\phi^4] = 4$

and hence, $\left(\mu\partial_\mu - \gamma\phi\frac{\partial}{\partial\phi}\right)\phi/\mu = -(1+\gamma)\,\phi/\mu$,

$$\left(\phi\frac{\partial}{\partial\phi} - \frac{\beta}{1+\gamma}\partial_\lambda + \frac{4\gamma}{1+\gamma}\right)u = 0 \tag{8.22}$$

with the solution

$$u(\phi/\mu, \lambda) = u_0(\bar\lambda)\, e^{-\int_\mu^\phi \frac{d\phi}{\phi}\frac{4\gamma}{1+\gamma}\bar\lambda(\phi)} \tag{8.23}$$

$$(\phi/\mu)\frac{\partial\bar\lambda}{\partial\,\phi/\mu} = \frac{\beta(\bar\lambda)}{1+\gamma(\bar\lambda)}$$

and in summary leading order in λ: $\gamma \simeq 0$ + 2-loop

$$V_{eff}(\phi) = u_0(\bar\lambda(\phi)) \cdot \phi^4 \qquad (8.24)$$

with
$$\bar\lambda(\phi) = \frac{\lambda}{1 - \lambda/8\pi^2 \, (N+8) \ln \phi/\mu} \qquad (8.25)$$

By comparing eq. (8.24) with (8.19) up to the 3rd order in λ we arrive at

$$V_{eff}(\phi) = \frac{1}{4}\phi^4\left(\bar\lambda + \frac{\bar\lambda^2}{(4\pi)^2}(N+8)\left(\ln\bar\lambda - \tfrac{3}{2}\right) + 9\ln 3\right) \qquad (8.26)$$

This looks similar to eq. (8.19) but the pot. large logs for $\phi \to 0$ have disappeared, as then $\bar\lambda \to 0$. This procedure is called an <u>RG-improvement</u>.