8.2 Anomalies

We consider a fermionic theory with massless fermions,

$$S[A, \psi, \bar\psi] = -\int d^4x \, \bar\psi \not{D} \psi \qquad (8.27)$$

coupled to a $U(1)$ gauge field. The action in eq. (8.27) is invariant under axial transformations

$$\psi \to e^{i\alpha\gamma_5}\psi = \psi^\alpha \qquad (8.28)$$

leading to the Noether current

$$j_{5\mu} = \bar\psi \gamma_\mu \gamma_5 \psi \quad \text{with} \quad \boxed{\partial_\mu j_{5\mu} = 0} \qquad (8.29)$$

How does eq. (8.29) look after fermionic quantisation?

To that end, let us consider

$$\int d^4x \, \langle \partial_\mu j_{5\mu} \rangle \qquad (8.30)$$

We compute first in the presence of a small mass: $\bar{\Psi}\not{\partial}\Psi \to \bar{\Psi}(\not{\partial}+m)\Psi$

$$\int d^4x\, \partial_\mu \langle \bar{\Psi}\gamma_\mu\gamma_5\Psi\rangle = 2m\int d^4x \langle\bar{\Psi}\gamma_5\Psi\rangle \tag{8.31}$$

The rhs of eq. (8.31) can be computed as

$$2m\int d^4x \langle\bar{\Psi}\gamma_5\Psi\rangle = -2m\int d^4x\, \mathrm{tr}\,\gamma_5 \underbrace{\langle\Psi\bar{\Psi}\rangle}_{\frac{\delta}{\delta\bar{\eta}}\frac{\delta}{\delta\eta}Z}$$

$$Z[\eta,\bar{\eta}] \simeq e^{\int\bar{\eta}\frac{1}{\not{\partial}+m}\eta} \longrightarrow \quad = 2m\int d^4x\, \mathrm{tr}\, \gamma_5\frac{1}{\not{\partial}+m} \tag{8.32}$$

The integral and Dirac trace can be rewritten in a sum of eigenvalues of $\not{\partial}$:

$$\not{\partial}\,|\varphi_n\rangle = \lambda_n|\varphi_n\rangle$$

$$\Rightarrow \not{\partial}\gamma_5|\varphi_n\rangle = -\lambda_n\gamma_5|\varphi_n\rangle \tag{8.33}$$

non-zero EV ~~are~~ paired.

We conclude

$$2m\int d^4x\, \mathrm{tr}\,\gamma_5\frac{1}{\not{\partial}+m} = 2m\sum_n \langle\varphi_n|\gamma_5|\varphi_n\rangle\frac{1}{\lambda_n+m}$$

$$= \frac{2m}{m}\sum_{n_0}\langle\varphi_{n_0}|\gamma_5|\varphi_{n_0}\rangle \tag{8.34}$$

$\langle\varphi_n|\gamma_5|\varphi_n\rangle = 0$ for $\lambda_n \neq 0$ via eq. (8.33) where $\not{D}\,|\varphi_{n0}\rangle = 0$, $|\varphi_{n0}\rangle$ are the zero modes. The φ_{n0} can be split into positive and negative chirality modes,

$$\gamma_5\,|\varphi_{n0}\rangle = \pm\,|\varphi_{n0}\rangle, \qquad (8.35)$$

as $\not{D}$ commutes with γ_5 on the zero mode space.

We conclude

$$\boxed{2m \int d^4x \;\mathrm{tr}\,\gamma_5 \frac{1}{\not{D}+m} = 2(N_+ - N_-),} \qquad (8.36)$$

where N_+, N_- are the number of zeromodes with pos. and neg. chirality respectively: right-handed and left-handed, $\frac{(1\pm\gamma_5)}{2}|\varphi_{n0\pm}\rangle = \pm\,|\varphi_{n0\pm}\rangle$ (8.37)

This leaves us with the puzzling result that the rhs of eq. (8.32) is non-zero, indeed it is the so-called (analytic) index of $\not{D}$, whereas the lhs is a total deriv..

Since the fermion is massive, we certainly have

$$\int d^4x \; \partial_\mu \langle \bar{\Psi} \gamma_5 \gamma_\mu \Psi \rangle = 0 \qquad (8.38)$$

What has gone wrong? We have implicitly used that

$$\int \mathcal{D}\Psi^\alpha \mathcal{D}\bar{\Psi}^\alpha = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \, J^{-1}(\alpha) \qquad (8.39)$$

with $J(\alpha) = 1$, as the transformation (8.28) is unitary.

Careful computation:

$$\int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} = \int \prod_n d a_n \prod_n d \bar{b}_n \qquad (8.40)$$

with

$$\Psi(x) = \sum a_n \varphi_n(x)$$
$$\bar{\Psi}(x) = \sum \bar{b}_n \varphi_n^+(x)$$

and $\not{D} \varphi_n(x) = \lambda_n \varphi_n(x)$ $\qquad (8.41)$

We have

$$\Psi^\alpha(x) = \sum a_n e^{i\alpha(x)\gamma_5} \varphi_n(x)$$
$$\bar{\Psi}^\alpha(x) = \sum \bar{b}_n \varphi_n^+(x) e^{i\alpha(x)\gamma_5} \qquad (8.42)$$

The new coefficient a_n^α is

$$a_n^\alpha = \int d^4x\, \varphi_n^\dagger(x)\, \psi^\alpha(x) = \int d^4x \sum_m a_m\, \varphi_n^\dagger(x)\, e^{i\alpha(x)\gamma_5} \varphi_m(x)$$
$$= a_n + i\sum_m \int d^4x\, \alpha(x)\, \varphi_n^\dagger(x)\, \gamma_5\, \varphi_m(x) + \mathcal{O}(\alpha(x)^2) \qquad (8.43)$$

We conclude that in leading order (linear) in $\alpha(x)$, the Jacobian J is given by

$$J(\alpha) = e^{2i \sum_n \int d^4x\, \alpha(x)\, \varphi_n^\dagger(x)\, \gamma_5\, \varphi_m(x)} \qquad (8.44)$$

It is left to compute

$$\sum_n \varphi_n^\dagger(x)\, \gamma_5\, \varphi_n(x) \cdot \left[e^{-\varepsilon\lambda_n^2} \right] \qquad (8.45)$$

We regularise eq. (8.45) with $e^{\varepsilon \not{D}^2}$ and get

$$\sum_n \varphi_n^\dagger(x)\, \gamma_5\, e^{\varepsilon \not{D}^2} \varphi_n(x)$$

$\langle x | \mathrm{tr}\, e^{\varepsilon \not{D}^2} | x \rangle$

$$= \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\, \gamma_5\, e^{-\varepsilon\left((p + gA_\nu + ig A_\nu \partial_\nu)^2 + \frac{i}{2}\sigma_{\mu\nu} F_{\mu\nu} \right)}$$

$\frac{1}{2i}[\gamma_\mu, \gamma_\nu]$

$$= -\frac{1}{8} \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\, \gamma_5\, \sigma_{\mu\nu}\, \sigma_{\rho\sigma}\, F_{\mu\nu} F_{\rho\sigma}\, e^{-\varepsilon p^2} + \mathcal{O}(\varepsilon) \qquad (8.46)$$

$$\boxed{\Rightarrow \sum_n \varphi_n^+(x)\,\gamma_5\, e^{\varepsilon \not{D}^2}\varphi_n(x) = -\frac{1}{2}\frac{1}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}} \tag{8.47}$$

and the Jacobian reads

$$J(\alpha) = e^{-i\frac{1}{16\pi^2}\int d^4x\, \alpha(x)\, \varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}} \tag{8.48}$$

In summary we conclude

$$\partial_\mu j_{5\mu} - 2m\langle\bar\psi\gamma_5\psi\rangle = -\frac{1}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{8.49}$$

and in the chiral limit

$$\boxed{\partial_\mu j_{5\mu} = -\frac{1}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}} \tag{8.50}$$

Eqs. (8.49), (8.50) imply the (baby) Atiyah-Singer index theorem:

$$\boxed{-\frac{1}{32\pi^2}\int \underbrace{\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}}_{\text{topol. invariant}} = \underbrace{N_+ - N_-}_{\text{analyt. index}}} \tag{8.51}$$