Quantum Field Theory 2 – Problem set 2

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Suggested reading before solving these problems: Chapter 1 in the script and/or chapter 9.1-9.3 in *Peskin & Schroeder*.

Problem 1: Wick's theorem reloaded

Using the generating functional for the free theory we can write:

$$\langle T\phi(x_1)\cdots\phi(x_n)\rangle = \frac{\delta}{i\delta j(x_1)}\cdots\frac{\delta}{i\delta j(x_n)} e^{-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y j(x)D(x-y)j(y)}\Big|_{j=0}$$

Use this expression to prove Wick's theorem, i.e. show that

$$\langle T\phi(x_1)\cdots\phi(x_n)\rangle = D(x_1-x_2)D(x_2-x_3)\cdots D(x_{n-1}-x_n) + \text{ all other correction.}$$

Problem 2: Feynman rules for a real ϕ^4-theory

Consider the generating functional for a real scalar field ϕ:

$$Z[j] = \int \mathcal{D}\phi \exp\left\{ i\int \mathrm{d}^4x\ [\mathcal{L}(\phi,\partial_\mu\phi) + j(x)\phi(x)]\right\}$$

with the Lagrangian density

$$\mathcal{L}(\phi,\partial_\mu\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(m^2 - i\epsilon)\phi^2 - \frac{\lambda}{4!}\phi^4$$

Derive the Feynman rules for this theory including the numerical factors from the path integral. Determine the symmetry factors for the diagrams contributing to the two-point function $\langle T\phi(x)\phi(y)\rangle$ up to $\mathcal{O}(\lambda^2)$.

(Optional) Problem 3: Quantum statistical mechanics

a) Consider the quantum statistical partition function of the canonical ensemble

$$Z = \mathrm{Tr}\, e^{-\beta H} \quad \text{where} \quad \beta = \frac{1}{T},$$

where H is the Hamiltonian. Use the same strategy that led to the path integral formula for matrix elements of e^{-iHt} in terms of the Lagrangian to derive a similar formula for Z. Show that one has to integrate over functions that are periodic in the "time argument" τ with range from 0 to $1/T$. Note that an Euclidean version of the action

$$S_E = \int_0^{1/T} d\tau L_E$$

appears in the weight.

b) Consider now a one-dimensional harmonic oscillator in an electric field. The Hamiltonian is

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 - eEx.$$

Show that the Euclidean action appearing in the path integral formula reads

$$S_E = \int_0^{1/T} d\tau \left\{ \frac{1}{2}m\,\dot{x}(\tau)^2 + \frac{1}{2}m\omega^2 x(\tau)^2 - eE\,x(\tau) \right\}.$$

c) Use a Fourier decomposition

$$x(\tau) = T\sum_{n=-\infty}^{\infty} x_n e^{i\omega_n \tau}, \qquad \text{with} \quad \omega_n = 2\pi T n, \quad x_n = \int_0^{1/T} d\tau e^{-i\omega_n\tau} x(\tau),$$

to rewrite S_E in terms of the fields x_n.

d) By performing now the path integral show that

$$Z = c\, e^{\frac{(eE)^2}{2mT\omega^2}},$$

where c is independent of the electric field E, and derive an expression for the susceptibility $\chi = \frac{\partial\langle e\,x\rangle}{\partial E}$.

e) Generalize now the construction of part a) to field theory. Derive an expression for the quantum statistical partition function of a scalar field in terms of a functional integral. Show for a free theory that the value of this integral is proportional to the formal expression

$$\left[\det(-\partial^2 + m^2)\right]^{-1/2},$$

where the operator acts on functions in Euclidean space that are periodic in the time direction with periodicity β.