Quantum Field Theory 2 – Problem set 11

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Suggested reading before solving these problems: Chapter 6.1 & 6.2 in the script and/or chapter 3 in *Rothe: Lattice Gauge Theories.*

Problem 1: Discrete derivatives

Consider a smooth function $f(x)$. Assume that for some reason you have access to this function only on a discrete set of points, say x, $x \pm a$, $x \pm 2a$, $x \pm 3a, \ldots$. Prove the following relations

$$\begin{aligned}
\frac{f(x+a)-f(x)}{a} = \frac{f(x)-f(x-a)}{a} &= f'(x) + \mathcal{O}(a), \\
\frac{f(x+a)-f(x-a)}{2a} &= f'(x) + \mathcal{O}(a^2), \\
\frac{f(x+a)+f(x-a)-2f(x)}{a^2} &= f''(x) + \mathcal{O}(a^2).
\end{aligned}$$

By using more points, find expressions for $f''(x) + \mathcal{O}(a^3)$ and $f' + \mathcal{O}(a^3)$.

Problem 2: Scalar field on the lattice

Consider the action for a free scalar field in Euclidean space

$$S = \int d^4x \frac{1}{2}\phi(x)\left(-\partial_\mu\partial_\mu + M^2\right)\phi(x).$$

A lattice formulation is obtained by making the following substitutions

$$\begin{aligned}
x &\to na \qquad \text{with} \qquad n = (n_1, n_2, n_3, n_4)\,, \\
\phi(x) &\to \phi_n = \phi(na)\,, \\
\int d^4x &\to a^4 \sum_n\,, \\
\partial_\mu\partial_\mu\phi(x) &\to \frac{1}{a^2}\widehat{\Delta}\phi(na)\,.
\end{aligned}$$

a) Convince yourself that a useful choice for the discrete Laplace operator is

$$\widehat{\Delta}\phi_n = \sum_{\mu=1,2,3,4}\left\{\phi_{n+e_\mu} + \phi_{n-e_\mu} - 2\phi_n\right\}\,,$$

and that the action becomes with this choice

$$S = -\frac{a^2}{2}\sum_{n,\mu}\phi_n\phi_{n+e_\mu} + \frac{a^2}{2}(8+a^2M^2)\sum_n \phi_n\phi_n\,.$$

What is the summation range of μ?

b) This action can be written in the form

$$S = \frac{a^4}{2}\sum_{n,m}\phi_n K_{nm}\phi_m.$$

Derive an explicit expression for the matrix K_{nm}.

c) Use the following representation of the Kronecker delta

$$\frac{1}{a^4}\delta_{nm} = \int_{-\pi/a}^{\pi/a}\frac{d^4q}{(2\pi)^4}e^{iq\cdot(n-m)a}\,,$$

to obtain the inverse propagator in momentum space which is defined by

$$K_{nm} = \int_{-\pi/a}^{\pi/a}\frac{d^4q}{(2\pi)^4}K(q)e^{iq\cdot(n-m)a}\,.$$

Problem 3 :Gauge invariance

In the lecture we have introduced the link variables $U_\mu(n)\in SU(N)$ with the transformation properties

$$U_\mu(n) \overset{G}{\to} G(n)U_\mu(n)G^\dagger(n+\hat\mu)\,.$$

a) Show that

$$\hat\phi_n^\dagger U_\mu(n)\hat\phi_{n+\hat\mu}$$

is gauge invariant with the complex scalar field $\hat\phi$ having the transformation properties

$$\hat\phi_n \overset{G}{\to} G(n)\hat\phi_n, \qquad \hat\phi_n^\dagger \overset{G}{\to} \hat\phi_n^\dagger G_n^\dagger\,.$$

b) Follow the steps is the lecture and show that the scalar action

$$S[\hat\phi,U] = -\sum_{n,\mu>0}\left(\hat\phi_n^\dagger U_\mu(n)\hat\phi_{n+\hat\mu} + \phi_n^\dagger U_\mu^\dagger(n-\hat\mu)\hat\phi_{n-\hat\mu} - 2\hat\phi_n^\dagger\hat\phi_n\right) + \hat m^2\sum_n \hat\phi_n^\dagger\hat\phi_n\,,$$

is gauge invariant. Here $\hat m^2 = a^2m^2$.

c) Determine the continuum limit of $S[\hat\phi,U]$.