Quantum Field Theory 2 – Problem set 12

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Suggested reading before solving these problems: Chapter 6.2-6.4 in the script and/or chapter 6, 7, 11.1 in *Rothe: Lattice Gauge Theories*.

Problem 1: Strong coupling expansion in Lattice QCD

Consider $SU(3)$ Yang-Mills theory on the lattice. The action is given by

$$S = -\beta \sum_P S_P$$

where $\beta = \frac{6}{g_0^2}$ and

$$S_P = \frac{1}{6}\mathrm{Tr}(U_P + U_P^\dagger)$$

is the contribution associated with a plaquette P. The sum goes over all distinct plaquettes. U_P is the plaquette variable and can be expressed in terms of the link variables around the plaquette P as

$$U_P = U_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n).$$

The Wilson loop with spatial extend $\hat{R}$ and temporal extend $\hat{T}$ (in units of the lattice spacing a) can be expressed in terms of the link variables by the path ordered product

$$W_C[U] = \mathrm{Tr}\prod U_l$$

of link variables around the loop.

Consider now the expectation value

$$\langle W_C[U]\rangle = \frac{\int DU W_C[U] e^{-S[U]}}{\int DU e^{-S[U]}}.$$

Expand the exponential in powers of the coupling β

$$e^{-S[U]} = e^{\beta \sum_P S_P} = \prod_P \left[\sum_n \frac{\beta^n}{n!}(S_P)^n\right]$$

and use the following integration rules for the integrals over the link variables $U \in SU(3)$

$$\int dU\, U^{ab} = 0,$$
$$\int dU\, U^{ab} U^{cd} = 0,$$
$$\int dU\, U^{ab} (U^\dagger)^{cd} = \frac{1}{3}\delta_{ad}\delta_{bc},$$
$$\int dU\, 1 = 1,$$

to show

$$\langle W_C[U] \rangle \sim \left(\frac{\beta}{c}\right)^{\hat{R}\hat{T}}$$

with some coefficient c.

The Wilson loop behaves for large $\hat{T}$ as

$$\langle W_C[U] \rangle \overset{\hat{T}\to\infty}{\to} F(\hat{R}) e^{-\hat{V}(\hat{R})\hat{T}}$$

so that you can deduce that the potential is linear in $\hat{R}$:

$$\hat{V}(\hat{R}) = \hat{\sigma}\hat{R},$$

with $\hat{\sigma} = -\ln(\beta/c)$ the string tension.