Quantum Field Theory 2 – Problem set 13

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Suggested reading before solving these problems: Chapter 7.1, 7.2 & 7.3 in the script.

Problem 1: Yang-Mills two loop beta function & Banks-Zaks fixed point

In the lectures has been show the calculation of the 1-loop beta function of QCD:

$$\beta(g)\Big|_{\text{1-loop}} = \frac{g^3}{16\pi^2}\left[-\frac{11}{3}N_c + \frac{2}{3}N_f\right] = -\beta_0 g^3.$$

Consider the particular case of $SU(3)$ Yang-Mills, with zero number of fermion $N_f = 0$ and three colors $N_c = 3$; for this theory the beta function is $\beta_0 = \frac{11}{16\pi^2}$. It is possible to compute the running coupling at 2-loop evaluating higher order diagrams in the perturbative expansion, the resulting β function is:

$$\beta(g)\Big|_{\text{2-loop}} = -\beta_0 g^3 - \beta_1 g^5 + \mathcal{O}(g^7),$$

with

$$\beta_0 = \frac{11}{16\pi^2}, \qquad \beta_1 = \frac{102}{(16\pi^2)^2}.$$

a) Show that the 2-loop contribution do not change qualitatively the running of the coupling.

Suppose that due to the interaction with further fields the overall sign of β_1 changed:

$$\beta(g)\Big|_{\text{2-loop}} = -\beta_0 g^3 - \beta_1 g^5 + \mathcal{O}(g^7)\,.$$

the general expression are:

$$\beta_0 = \frac{1}{16\pi^2}\left[\frac{11}{3}N_c - \frac{2}{3}N_f\right], \quad \beta_1 = \frac{1}{(16\pi^2)^2}\left[\frac{34}{3}N_c^2 - N_f\left(\frac{N_c^2-1}{N_c} + \frac{20}{6}N_c\right)\right]$$

in particular with 3 colors $N_c = 3$ and 9 flavors $N_f = 9$ we have:

$$\beta_0 = \frac{5}{16\pi^2}, \quad \beta_1 = -\frac{3}{64\pi^4}.$$

b) Show that this theory has a fixed point, i.e. a value of the coupling g_* where $\beta(g_*) = 0$.

c) Study the stability of this fixed point, linearizing the flow equation around g_*.

d) What is the ultraviolet and infrared behavior of this theory?

Problem 2: Asymptotic safety

Consider a theory with two coupling g and λ with the following flow equation:

$$\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} = (2+\eta_N)g$$
$$\mu\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = -(2-\eta_N)\lambda + g\frac{A_1 + g(A_1B_2 - A_2B_1)}{2(1+B_2 g)}$$

where

$$A_1 = \frac{1+8\lambda}{\pi(1-2\lambda)}, \qquad A_2 = \frac{5}{6\pi(1-2\lambda)},$$

$$\eta_N = \frac{gB_1}{1+gB_2}, \quad \text{with} \quad B_1 = \frac{-11+18\lambda-28\lambda^2}{3\pi(1-2\lambda)^2}, \quad B_2 = -\frac{1+10\lambda}{12\pi(1-2\lambda)^2}.$$

This set of flow equation exhibits a non-Gaußian fixed point in the region $\lambda \geq 0$ and $g \geq 0$, i.e. a point in the parameter space where $\beta_g(g_*, \lambda_*) = 0$ and $\beta_\lambda(g_*, \lambda_*) = 0$ simultaneously.

a) Find the fixed points (g_*, λ_*) numerically.

b) Study the eigenvalues of stability matrix around the fixed points

$$-\frac{\partial\beta_i}{\partial\lambda_j}\Big|_{g=g_*,\lambda=\lambda_*},$$

with $\beta_i = (\beta_g, \beta_\lambda)$, $\lambda_i = (g, \lambda)$. Convince yourself that this fixed point is UV-stable. Is the perturbative (Gaußian) fixed point UV stable?

This flow equation correspond to the Einstein-Hilbert truncation:

$$\Gamma = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{g}[-R+2\Lambda] + \text{gauge-fixing terms} \tag{1}$$

where R is the Ricci scalar, G_N is the Newton constant and Λ is the value of the cosmological constant (not to be confused with the cutoff scale). The previous coupling are the dimensionless version of this two, i.e.:

$$g = \mu^2 G_N, \quad \lambda = \mu^{-2}\Lambda\,.$$

This analysis indicates the potential existence of quantum gravity as a non-perturbative quantum field theory.