Quantum Field Theory 2 – Tutorial 1

Lectures: Jan Pawlowski   pawlowski@thphys.uni-heidelberg.de

Tutorials: Eduardo Grossi   grossi@thphys.uni-heidelberg.de

Institut für Theoretische Physik, Uni Heidelberg   tutorial date: 18 April 2017

Problem 1: One dimensional Gaussian integrals

a) Prove the following identity for $a \in \mathbb{C}$, $\mathrm{Re}(a) > 0$

$$\int_{-\infty}^{\infty} dx \, e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}.$$

b) Show the generalization

$$\int_{-\infty}^{\infty} dx \, e^{-\frac{a}{2}x^2 + bx} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}.$$

c) Consider now Gaussian integrals over complex numbers $z = \frac{1}{\sqrt{2}}(x + iy)$ with $\int dz dz^* = \int dx dy$. Show for $w \in \mathbb{C}$, $\mathrm{Re}(w) > 0$

$$\int dz dz^* \, e^{-z^* wz} = \frac{2\pi}{w},$$

as well as the generalization for arbitrary $u, v \in \mathbb{C}$

$$\int dz dz^* \, e^{-z^* wz + u^* z + z^* v} = \frac{2\pi}{w} e^{u^* w^{-1} v}.$$

Problem 2: Gaussian integrals in more than one dimension

a) Proof for a real, symmetric and positive definite N-dimensional matrix $\mathbf{A}$ and $\mathbf{x} \in \mathbb{R}^N$ a real N-component vector

$$\int d\mathbf{x} \, e^{-\frac{1}{2}\mathbf{x}^T \mathbf{A} \mathbf{x}} = (2\pi)^{N/2} \frac{1}{\sqrt{\det \mathbf{A}}}.$$

(Tip: Use a change of integration variables to diagonalize $\mathbf{A}$.)

b) Proof the generalization

$$\int d\mathbf{x} \, e^{-\frac{1}{2}\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{j}^T \mathbf{x}} = (2\pi)^{N/2} \frac{1}{\sqrt{\det \mathbf{A}}} \, e^{\frac{1}{2}\mathbf{j}^T \mathbf{A}^{-1} \mathbf{j}}. \tag{1}$$

c) Show for hermitian and positive $\mathbf{A} = \mathbf{A}^\dagger$

$$\int d\mathbf{z} d\mathbf{z}^\dagger \, e^{-\mathbf{z}^\dagger \mathbf{A} \mathbf{z} + \mathbf{u}^\dagger \mathbf{z} + \mathbf{z}^\dagger \mathbf{v}} = \frac{(2\pi)^N}{\det \mathbf{A}} \, e^{\mathbf{u}^\dagger \mathbf{A}^{-1} \mathbf{v}}.$$

Problem 3: Expectation values and Wick's theorem

a) Consider expectation values with respect to the Gaussian weight

$$\langle \ldots \rangle = \frac{\int d\mathbf{x} \, (\ldots) \, e^{-\frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}}}{\int d\mathbf{x} \, e^{-\frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}}}.$$

Using Eq. (1) show

$$
\begin{aligned}
\langle x_m \rangle &= 0, \\
\langle x_m x_n \rangle &= A^{-1}_{mn}, \\
\langle x_m x_n x_p \rangle &= 0, \\
\langle x_m x_n x_p x_q \rangle &= A^{-1}_{mn}A^{-1}_{pq} + A^{-1}_{mp}A^{-1}_{nq} + A^{-1}_{mq}A^{-1}_{np},
\end{aligned}
$$

and more general

$$\langle x_{i_1} x_{i_2} \ldots x_{i_n} \rangle = \begin{cases} 0 & n \text{ odd} \\ \text{all full contractions} & n \text{ even.} \end{cases}$$

This is a version of Wick's theorem for real bosonic fields.

b) Derive the corresponding relations for complex Gaussian integrals.