# Quantum Field Theory 2 – Problem set 1


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Suggested reading before solving the problems below: Chapter 1 in the script and/or Sections 9.1-9.3 in *Peskin & Schroeder*.

### Problem 1: Asymptotic series

Consider the following function

$$Z(\lambda) = \int_{-\infty}^{\infty} d\varphi \; e^{-\frac{1}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4}, \tag{1}$$

with $\lambda$ being a non-negative real variable.

a) Calculate the coefficients $Z_n$ of the perturbative series in $\lambda$

$$Z(\lambda) = \sum_{n=0}^{\infty} Z_n \lambda^n. \tag{2}$$

**Hint:** You may use the definition $\Gamma(x+1) = \int_0^\infty dt\; e^{-t} t^x$ of the gamma function.

b) The error of the partial sum of order $N$ can can be estimated by an upper bound according to

$$\begin{aligned} R_N &= \left| Z(\lambda) - \sum_{n=0}^{N} Z_n \lambda^n \right| \\ &\leq \int d\varphi \; e^{-\frac{1}{2}\varphi^2} \left| e^{-\frac{1}{4}\lambda\varphi^4} - \sum_{n=0}^{N} \frac{1}{n!} \left( -\frac{1}{4}\lambda\varphi^4 \right)^n \right| \\ &\leq \int d\varphi \; e^{-\frac{1}{2}\varphi^2} \frac{1}{(N+1)!} \left( \frac{1}{4}\lambda\varphi^4 \right)^{N+1} = \lambda^{N+1} |Z_{N+1}|. \end{aligned} \tag{3}$$

Use the Stirling's formula

$$n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \left( 1 + O\left( \frac{1}{n} \right) \right) \tag{4}$$

to estimate the quantity $\lambda^n Z_n$ for large $n$ and determine the (approximate) order $N = N_{\min}$ for which $R_N$ (rather its upper limit found above) is minimal.

**Problem 2: Generating functions**

Consider a probabilistic theory for an $N$-dimensional vector $\boldsymbol{x}$. The expectation value $\langle x_{i_1} \cdots x_{i_L} \rangle$ of the generic product $x_{a_1} \cdots \cdot x_{a_n}$ is defined as

$$\langle x_{i_1} \cdots x_{i_L} \rangle = \frac{\int d^n x \, x_{i_1} \cdots x_{i_L} e^{-S(\boldsymbol{x})}}{\int d\boldsymbol{x} \; e^{-S(\boldsymbol{x})}}. \tag{5}$$

where the "action" $S(\boldsymbol{x})$ is defined

$$S(\boldsymbol{x}) = \frac{1}{2} P_{ab} \, x_a x_b + \frac{1}{3!} \gamma_{abc} \, x_a x_b x_c + \frac{1}{4!} \lambda_{abcd} \, x_a x_b x_c x_d + \cdots \tag{6}$$

in terms of the real quantities $P_{ab}$, $\gamma_{abc}$, $\lambda_{abcd}$ and so on (summation over repeated indexes is implied), which are such that the integrals are convergent and, below, we also assume that the matrix $P_{ab}$ is invertible.

It is useful to introduce the partition function $Z(\boldsymbol{J})$ of the $N$-dimensional current $\boldsymbol{J}$ as

$$Z(\boldsymbol{J}) = \int d\boldsymbol{x} \; e^{-S(\boldsymbol{x}) + J_a x_a}. \tag{7}$$

a) Show that

$$\langle x_{i_1} \ldots x_{i_L} \rangle = \frac{1}{Z(\boldsymbol{J})} \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_L}} Z(\boldsymbol{J}) \bigg|_{\boldsymbol{J}=\boldsymbol{0}}. \tag{8}$$

b) Show that one can formally write

$$Z(\boldsymbol{J}) = \sqrt{\frac{(2\pi)^N}{\det P}} \; e^{-\left( \frac{1}{3!} \gamma_{abc} \frac{\partial}{\partial J_a} \frac{\partial}{\partial J_b} \frac{\partial}{\partial J_c} + \frac{1}{4!} \lambda_{abcd} \frac{\partial}{\partial J_a} \frac{\partial}{\partial J_b} \frac{\partial}{\partial J_c} \frac{\partial}{\partial J_d} + \cdots \right)} \; e^{\frac{1}{2} J_a P^{-1}_{ab} J_b}. \tag{9}$$

c) Derive the expressions of the expectations values or "correlation functions" $\langle x_a \rangle$, $\langle x_a x_b \rangle$, $\langle x_a x_b x_c \rangle$, and $\langle x_a x_b x_c x_d \rangle$ up to linear terms in $\gamma_{abc}$ and $\lambda_{abcd}$. Can you draw corresponding Feynman diagrams?

**Note:** Ignore the higher-order terms in $S(\boldsymbol{x})$ corresponding to the dots in Eq. (6).

d) The connected correlation functions are given by

$$\langle x_{i_1} \ldots x_{i_L} \rangle_c = \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_L}} W(\boldsymbol{J}) \bigg|_{\boldsymbol{J}=\boldsymbol{0}}, \tag{10}$$

with

$$W(\boldsymbol{J}) = \ln Z(\boldsymbol{J}). \tag{11}$$

Derive expressions for $\langle x_a \rangle_c$, $\langle x_a x_b \rangle_c$, $\langle x_a x_b x_c \rangle_c$, and $\langle x_a x_b x_c x_d \rangle_c$ up to linear terms in $\gamma_{abc}$ and $\lambda_{abcd}$.

**Note:** The generating function $W(\boldsymbol{J})$ of the connected correlation functions is also known as the Schwinger function.