Quantum Field Theory 2 – Problem set 2

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Suggested reading before solving these problems: Chapter 1 in the script and/or chapter 9.1-9.3 in *Peskin & Schroeder*.

Problem 1: Wick's theorem reloaded

By employing the generating functional

$$Z_0[j] = \int \mathcal{D}\phi \exp\left\{ i \int \mathrm{d}^4x \, \left[\mathcal{L}_0(\phi, \partial_\mu \phi) + j(x)\phi(x)\right] \right\}, \tag{1}$$

with the Lagrangian density

$$\mathcal{L}_0(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \tag{2}$$

of the free theory of a real scalar field $\phi(x)$, we can write:

$$\langle T\phi(x_1)\cdots\phi(x_n)\rangle = \frac{\delta}{i\delta j(x_1)} \cdots \frac{\delta}{i\delta j(x_n)} e^{-\frac{1}{2}\int \mathrm{d}^4x \mathrm{d}^4y j(x) D(x-y) j(y)} \Big|_{j=0} \tag{3}$$

where $D(x-y)$ is the free propagator. Use this expression to prove Wick's theorem, i.e., show that for odd n the correlator $\langle T\phi(x_1)\cdots\phi(x_n)\rangle$ vanishes, whereas for even n

$$\langle T\phi(x_1)\cdots\phi(x_n)\rangle = D(x_1 - x_2) D(x_3 - x_4) \cdots D(x_{2k-1} - x_{2k}) + \cdots , \tag{4}$$

where $k = n/2$ and the dots indicate the sum over all possible remaining contractions such that the right-hand-side is symmetric under the exchange of any two indexes like the left-hand side (there are $1 \times 3 \times \cdots \times (2k-1) = (2k)!/(2^k k!)$ terms altogether).

Problem 2: Feynman rules for real scalar field

Consider the generating functional for a real scalar field $\phi(x)$:

$$Z[j] = \int \mathcal{D}\phi \exp\left\{ i \int \mathrm{d}^4x \, \left[\mathcal{L}(\phi, \partial_\mu \phi) + j(x)\phi(x)\right] \right\} \tag{5}$$

with the Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ being given by

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4. \tag{6}$$

Derive the Feynman rules for this theory including the numerical factors from the path integral. Determine the symmetry factors for the diagrams contributing to the two-point function $\langle T\phi(x)\phi(y)\rangle$ up to $\mathcal{O}(\lambda^2)$.

Problem 3 (Optional): Quantum statistical mechanics

a) Consider the quantum statistical partition function of the canonical ensemble

$$Z = \mathrm{Tr}\, e^{-\beta H}, \tag{7}$$

where $\beta = 1/T$ is the inverse of the temperature T and H is the Hamiltonian. Use the same strategy that led to the path integral formula for matrix elements of e^{-iHt} in terms of the Lagrangian to derive a similar formula for Z. Show that one has to integrate over functions that are periodic in the "time argument" τ with range from 0 to $1/T$. Note that a Euclidean version of the action

$$S_E = \int_0^{1/T} d\tau L_E \tag{8}$$

appears in the weight, with L_E being interpreted as the Euclidean version of the Lagrangian.

b) Consider now a one-dimensional harmonic oscillator of mass m and charge e in the presence a constant electric field E. The Hamiltonian is

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 - eEx. \tag{9}$$

Show that the Euclidean action appearing in the path integral formula reads

$$S_E = \int_0^{1/T} d\tau \left\{ \frac{1}{2}m\,\dot{x}(\tau)^2 + \frac{1}{2}m\omega^2 x(\tau)^2 - eE\,x(\tau) \right\}. \tag{10}$$

c) Use the Fourier decomposition

$$x(\tau) = T\sum_{n=-\infty}^{\infty} x_n e^{i\omega_n\tau}, \qquad \text{with} \quad \omega_n = 2\pi T n, \quad x_n = \int_0^{1/T} d\tau e^{-i\omega_n\tau} x(\tau),$$

to rewrite S_E in terms of the amplitudes or "fields" x_n.

d) By computing now the path integral show that

$$Z = c\,\exp\left[\frac{(eE)^2}{2mT\omega^2}\right], \tag{11}$$

where c is independent of the electric field E, and derive an expression for the susceptibility $\chi = \partial\langle ex\rangle/\partial E$.

e) Generalize now the construction of part a) to field theory. Derive an expression for the quantum statistical partition function of a scalar field in terms of a functional integral. Show for a free theory that the value of this integral is proportional to the formal expression

$$\left[\det(-\partial^2 + m^2)\right]^{-1/2},$$

where the operator acts on functions in Euclidean space that are periodic in the time direction with periodicity β.