Quantum Field Theory 2 – Problem set 3

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Suggested reading before solving these problems: Chapter 1, 2.1 in the script and/or chapter 9.5, 11.3 –11.5 in *Peskin & Schroeder*.

Problem 1: Effective potential

In the lecture you derived an expression for the effective action in one-loop approximation

$$\Gamma_{\text{1-loop}}[\phi] = S[\phi] + \frac{1}{2}\mathrm{Tr}\ln S^{(2)}[\phi]. \tag{1}$$

Consider the ϕ^4-theory in d Euclidean dimensions

$$S[\phi] = \int d^d x \left\{ \frac{1}{2}\phi(-\partial_\mu\partial_\mu)\phi + V_0(\phi) \right\}, \tag{2}$$

with $V_0(\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{1}{4!}\lambda_0\phi^4$. Write the effective action as the derivative expansion

$$\Gamma_{\text{1-loop}}[\phi] = \int d^d x \left[\frac{1}{2}\phi(-Z_1\partial_\mu\partial_\mu + Z_2\partial_\mu\partial_\mu\partial_\nu\partial_\nu + \dots)\phi + V_{\text{eff}}(\phi) \right] + \text{const.} \tag{3}$$

The first term vanishes for constant field configurations, so that

$$\Gamma_{\text{1-loop}}[\phi]\bigg|_{\phi=\text{const}} = \int d^d x \; V_{\text{eff}}(\phi) + \text{const.}$$

The functional trace operation Tr is best evaluated in momentum space

$$\phi(x) = \int \frac{d^d p}{(2\pi)^d}\, \tilde{\phi}(p)\, e^{ipx}. \tag{4}$$

a) Show that in the case of a constant field ϕ the effective potential is given by

$$V_{\text{eff}}(\phi) = V_0(\phi) + \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \ln\left(p^2 + m_0^2 + \frac{\lambda_0}{2}\phi^2 \right) + \text{const.} \tag{5}$$

b) It is often useful to expand the effective potential in a Taylor series. A discrete symmetry $\phi \to -\phi$ implies that only even terms appear in the expansion:

$$V_{\text{eff}}(\phi) = V_{\text{eff}}(0) + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + \dots. \tag{6}$$

Derive the formal expressions of m^2 and λ.

c) The effective action $\Gamma[\phi]$ can be used to generate the amputated n-point functions. Calculate the amplitudes corresponding to the Feynman diagrams up to 1-loop order for the two- and the four-point functions at vanishing external momenta. Show that one indeed obtains the same expressions as derived in part b).

Problem 2: Grassmann variables: Change of variables

Consider a set of Grassmann generators $c_1, \ldots, c_n$ and a non-singular matrix A. For $c'_i = A_{ij} c_j$ prove that

$$c'_1 \ldots c'_n = (\det A) c_1 \ldots c_n.$$

Show that this implies

$$\int dc_1 \ldots dc_n = \int dc'_1 \ldots dc'_n \, J, \qquad \text{with} \qquad J = \det \frac{\partial c'_i}{\partial c_j}.$$