Quantum Field Theory 2 – Problem set 4

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Suggested reading before solving these problems: Chapter 2.2-3.1 in the script and/or chapter 11.3 –11.5 in *Peskin & Schroeder*.

Problem 1: Effective action

Consider a scalar field theory in d Euclidean dimensions. The partition function is given by

$$e^{W[J]} = Z[J] = \int d\varphi \; e^{-S[\varphi]+\int d^dx\, J(x)\varphi(x)}. \tag{1}$$

The effective action $\Gamma[\phi]$ is defined as the Legendre transform of the Schwinger functional $W[J]$:

$$\Gamma[\phi] = \sup_J \left(\int d^dx J(x)\phi(x) - W[J] \right). \tag{2}$$

a) Show that this definition implies

$$\phi(x) = \frac{\delta W[J]}{\delta J(x)}, \tag{3a}$$

$$J(x) = \frac{\delta \Gamma[\phi]}{\delta \phi(x)}, \tag{3b}$$

and

$$\int d^dz \Gamma^{(2)}(x,z) W^{(2)}(z,y) = \delta^{(d)}(x-y). \tag{4}$$

In a symbolic notation the last line reads

$$\Gamma^{(2)} = \left(W^{(2)}\right)^{-1}. \tag{5}$$

b) Show that the effective action can be written as an implicit functional integral in the presence of a “background field” ϕ

$$e^{-\Gamma[\phi]} = \int D\varphi \; \exp\left[-S[\phi+\varphi] + \int d^dx \; \frac{\delta\Gamma[\phi]}{\delta\phi(x)}\varphi(x) \right]. \tag{6}$$

c) Assume that the fluctuation field φ is a perturbation of the background field ϕ. In the lowest order (tree approximation) one has (up to an additive constant)

$$\Gamma[\phi] = S[\phi]. \tag{7}$$

At the second order one instead obtains

$$\begin{aligned} S[\phi+\varphi] \approx S[\phi] &+ \int d^d x\, \frac{\delta S[\phi]}{\delta\phi(x)} \varphi(x) \\ &+ \frac{1}{2}\int d^d x d^d y\, S^{(2)}[\phi](x,y)\, \varphi(x)\varphi(y). \end{aligned} \tag{8}$$

Show that this leads to the one-loop expression

$$\Gamma[\phi] = S[\phi] + \frac{1}{2}\,\mathrm{Tr}\ln S^{(2)}[\phi] + \ldots \tag{9}$$

d) Often one is interested in the expansion of $\Gamma[\phi]$ for small values of the field ϕ. One can then expand $S^{(2)}$

$$\begin{aligned} S^{(2)}[\phi] = S^{(2)}[0] &+ \int d^d x\, S^{(3)}[0](x)\, \phi(x) \\ &+ \frac{1}{2}\int d^d x d^d y\, S^{(4)}[0](x,y)\, \phi(x)\phi(y) + \ldots \end{aligned} \tag{10}$$

By expanding also the logarithm in Eq. (6) show that the effective action has the expansion

$$\Gamma[\phi] = S[\phi] + \Delta\Gamma^{(0)} + \int d^d x\, \Delta\Gamma^{(1)}(x)\phi(x) + \frac{1}{2}\int d^d x d^d y\, \Delta\Gamma^{(2)}(x,y)\phi(x)\phi(y) + \ldots, \tag{11}$$

with

$$\Delta\Gamma^{(0)} = \frac{1}{2}\,\mathrm{Tr}\left\{\ln S^{(2)}[0]\right\}, \tag{12a}$$

$$\Delta\Gamma^{(1)}(x) = \frac{1}{2}\,\mathrm{Tr}\left\{(S^{(2)}[0])^{-1} S^{(3)}[0](x)\right\}, \tag{12b}$$

and

$$\begin{aligned} \Delta\Gamma^{(2)}(x,y) = \frac{1}{2}\,\mathrm{Tr}\Big\{ &\left(S^{(2)}[0]\right)^{-1} S^{(4)}[0](x,y) \\ &- \left(S^{(2)}[0]\right)^{-1} S^{(3)}[0](x) \left(S^{(2)}[0]\right)^{-1} S^{(3)}[0](y)\Big\}. \end{aligned} \tag{13}$$

e) Can you interpret these expressions in terms of Feynman diagrams?
Hint: Remember which kind of Feynman diagrams are generated by the effective action.

Problem 2: Yukawa theory in d Euclidean dimensions

Consider the action of the Yukawa theory in d Euclidean dimensions:

$$S[\psi,\bar\psi,\varphi] = \int d^d x \left\{ -\bar\psi(\gamma^\mu\partial_\mu + m)\psi - h\bar\psi\psi\varphi + \frac{1}{2}\varphi(-\partial_\mu\partial_\mu + m_\varphi^2)\varphi \right\}. \tag{14}$$

Following the steps performed in the lecture, show that the boson two-point function has a contribution at one-loop level

$$\langle\varphi(-q)\varphi(p)\rangle_{1-\text{loop,conn.}} = (2\pi)^d\delta^{(d)}(p-q)G_\varphi(p)\Pi(p)G_\varphi(p), \tag{15}$$

with

$$\Pi(p) = -h^2\int\frac{d^dq}{(2\pi)^d}\frac{\text{tr}\left[(-i\gamma^\mu q_\mu + m)(-i\gamma^\nu(p_\nu + q_\nu) + m)\right]}{(q^2+m^2)((p+q)^2+m^2)}. \tag{16}$$

Using the identity $\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}$ show that

$$\Pi(p) = -h^2\int\frac{d^dp}{(2\pi)^d}\frac{-d\;q\cdot(q+p) + d\;m^2}{(q^2+m^2)((q+p)^2+m^2)}. \tag{17}$$