Quantum Field Theory 2 – Problem set 6

Lectures: Jan Pawlowski J.Pawlowski@thphys.uni-heidelberg.de
Tutorials: Aleksandr Mikheev A.Mikheev@thphys.uni-heidelberg.de
Institut für Theoretische Physik, Uni Heidelberg tutorial date: week of 22.05.2023

Suggested reading before solving these problems: Chapter 13.1 in the script and/or chapter 15 in *Peskin & Schroeder*.

Problem 1: Representations of Lie algebras

An element of a Lie group close to the identity can be written as

$$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}(\alpha^2), \tag{1}$$

where α^a are real parameters, Hermitian operators T^a are the generators of the Lie group, and the sum over the index a is implied. The generators of the Lie group satisfy the commutation relations

$$[T^a, T^b] = if^{abc}T^c, \tag{2}$$

with f^{abc} being the structure constants. The vector space spanned by the generators with the additional *Lie bracket* structure (2) is called *Lie algebra*.

a) Prove the identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0, \tag{3}$$

and that it implies

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0. \tag{4}$$

b) What is a representation t^a_r of a Lie algebra and what does "irreducible" mean?

c) Assume that the generators in some representation r are normalized as

$$\mathrm{tr}\{t^a_r t^b_r\} = C(r)\,\delta^{ab}. \tag{5}$$

Show that this yields the following representation of the structure constants

$$f^{abc} = -\frac{i}{C(r)}\,\mathrm{tr}\left\{\left[t^a_r, t^b_r\right] t^c_r\right\}, \tag{6}$$

and that f^{abc} is totally antisymmetric.

d) Consider now the group SU(N), with generators t_r^a. The fundamental representation is given by

$$\phi \to (1 + i\alpha^a t_r^a)\phi + O(\alpha^2), \tag{7}$$

where ϕ is an N-dimensional complex vector. Show that the matrices $t_{\bar{r}}^a = -(t_r^a)^*$ also lead to a representation (the *conjugate* representation).

e) Similarly, show that the matrices $(t_G^b)^{ac} = if^{abc}$ define a representation (the *adjoint* representation).

Problem 2: Field strength tensor

For a non-Abelian gauge theory, the covariant derivative is given by $D_\mu = \partial_\mu - igA_\mu^a t^a$. The field strength tensor F can be defined as

$$[D_\mu, D_\nu] = -igF_{\mu\nu}^a t^a. \tag{8}$$

a) Derive the more explicit form of the field strength tensor

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c. \tag{9}$$

b) How do the field A_μ^a and the field strength tensor $F_{\mu\nu}^a$ transform under infinitesimal and finite local gauge transformations?

c) Show that the quantity $F_{\mu\nu}^a F^{a\,\mu\nu}$ is gauge-invariant.