Quantum Field Theory 2 – Problem set 8

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Suggested reading before solving these problems: Chapter 13.4 in the script and/or chapter 15.7, 17.1 in *Weinberg, The Quantum Theory of Fields 2*.

Problem 1: BRST Symmetry and the effective action

Consider the generating functional of Yang-Mills theory

$$Z[J,\eta,\bar\eta] = \int DADcD\bar{c}Db\; e^{-S[A,c,\bar c,b]+\int_x\{J\cdot A+\bar\eta\cdot c-\bar c\cdot\eta\}}.$$

a) Use the symmetry of the action S and of the measure $DADcD\bar{c}Db$ under BRST transformations to derive

$$\int_x \{J\cdot\langle\epsilon QA\rangle + \bar\eta\cdot\langle\epsilon Qc\rangle - \langle\epsilon Q\bar c\rangle\cdot\eta\} = 0\,. \tag{1}$$

Why is it not possible to replace $\langle\epsilon QA\rangle$ by $\epsilon Q\langle A\rangle$ etc. in the above expression?

b) Introduce source terms for QA, Qc, and $Q\bar c$:

$$Z[J,\eta,\bar\eta,L_A,L_c,L_{\bar c}] = \int DADcD\bar{c}Db\; e^{-S[A,c,\bar c,b]+\int_x\{J\cdot A+\bar\eta\cdot c-\bar c\cdot\eta+L_A\cdot QA+L_c\cdot Qc+L_{\bar c}\cdot Q\bar c\}}\,. \tag{2}$$

Why does Eq. (1) still hold for arbitrary sources L_A, L_c, $L_{\bar c}$? Demonstrate that Eq. (1) can be rewritten as

$$\int_x \left\{ J\cdot\frac{\delta Z}{\delta L_A} - \bar\eta\cdot\frac{\delta Z}{\delta L_c} - \frac{\delta Z}{\delta L_{\bar c}}\cdot\eta \right\} = 0. \tag{3}$$

c) The effective action Γ is defined as the Legendre transform of $\ln Z$:

$$\Gamma[A,c,\bar c;L_A,L_c,L_{\bar c}] = \int_x \{J\cdot A+\bar\eta\cdot c-\bar c\cdot\eta\} - \ln Z[J,\eta,\bar\eta,L_A,L_c,L_{\bar c}], \tag{4}$$

where $J=\delta\Gamma/\delta A$, $\bar\eta = -\delta\Gamma/\delta c$, and $\eta = -\delta\Gamma/\delta\bar c$. Prove the relations

$$\frac{1}{Z}\frac{\delta Z}{\delta L_A} = -\frac{\delta\Gamma}{\delta L_A},\quad \frac{1}{Z}\frac{\delta Z}{\delta L_c} = -\frac{\delta\Gamma}{\delta L_c},\quad \frac{1}{Z}\frac{\delta Z}{\delta L_{\bar c}} = -\frac{\delta\Gamma}{\delta L_{\bar c}}. \tag{5}$$

d) Show that Eq. (3) leads to the quantum master equation for the effective action

$$\int_x \left\{ \frac{\delta\Gamma}{\delta L_A}\cdot\frac{\delta\Gamma}{\delta A} + \frac{\delta\Gamma}{\delta L_c}\cdot\frac{\delta\Gamma}{\delta c} + \frac{\delta\Gamma}{\delta L_{\bar c}}\cdot\frac{\delta\Gamma}{\delta\bar c} \right\} = 0\,. \tag{6}$$