Quantum Field Theory 2 – Problem set 11

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Suggested reading before solving these problems: Chapter 15.1 & 15.3 in the script and/or chapter 3 in *Rothe: Lattice Gauge Theories*.

Problem 1: Discrete derivatives

Consider a smooth function $f(x)$ and assume that for some reason you have access to this function only on a discrete set of points, say x, $x \pm a$, $x \pm 2a$, $x \pm 3a, \ldots$. Show that

$$\frac{f(x+a)-f(x)}{a} = \frac{f(x)-f(x-a)}{a} = f'(x) + \mathcal{O}(a), \tag{1a}$$

$$\frac{f(x+a)-f(x-a)}{2a} = f'(x) + \mathcal{O}(a^2), \tag{1b}$$

$$\frac{f(x+a)+f(x-a)-2f(x)}{a^2} = f''(x) + \mathcal{O}(a^2). \tag{1c}$$

By using more points, find expressions for $f''(x) + \mathcal{O}(a^3)$ and $f' + \mathcal{O}(a^3)$.

Problem 2: Scalar field on the lattice

Consider the action for a free scalar field in Euclidean space

$$S = \int d^4x \frac{1}{2}\phi(x)\left(-\partial_\mu\partial_\mu + M^2\right)\phi(x). \tag{2}$$

A lattice formulation is obtained by virtue of the following substitutions

$$\begin{aligned} x &\to na\,, \\ \phi(x) &\to \phi_n = \phi(na)\,, \\ \partial_\mu\partial_\mu\phi(x) &\to \frac{1}{a^2}\widehat{\Delta}\phi(na)\,, \\ \int d^4x &\to a^4\sum_n\,, \end{aligned} \tag{3}$$

with $n = (n_1, n_2, n_3, n_4)$.

a) Convince yourself that a useful choice for the discrete Laplace operator is

$$\widehat{\Delta}\phi_n = \sum_{\mu=1,\ldots,4}\left\{\phi_{n+e_\mu} + \phi_{n-e_\mu} - 2\phi_n\right\}\,, \tag{4}$$

and show that the lattice action with this choice reads

$$S = -\frac{a^2}{2}\sum_{n,\mu}\phi_n\phi_{n+e_\mu} + \frac{a^2}{2}(8+a^2M^2)\sum_n \phi_n\phi_n\,. \tag{5}$$

What is the summation range of μ in Eq. (5)?

b) The action Eq. (5) can be written in the form

$$S = \frac{a^4}{2}\sum_{n,m}\phi_n K_{nm}\phi_m. \tag{6}$$

Derive an explicit expression for the matrix K_{nm}.

c) Use the following representation of the Kronecker delta

$$\frac{1}{a^4}\delta_{nm} = \int_{-\pi/a}^{\pi/a}\frac{d^4q}{(2\pi)^4}e^{iq\cdot(n-m)a}\,, \tag{7}$$

to obtain the inverse propagator in momentum space, defined as

$$K_{nm} = \int_{-\pi/a}^{\pi/a}\frac{d^4q}{(2\pi)^4}K(q)\,e^{iq\cdot(n-m)a}\,. \tag{8}$$

Problem 3: Gauge invariance

In the lecture, we have introduced the link variables $U_\mu(n)\in SU(N)$ with the transformation properties

$$U_\mu(n)\xrightarrow{G} G(n)U_\mu(n)G^\dagger(n+\hat{\mu})\,.$$

a) Show that

$$\hat{\phi}_n^\dagger U_\mu(n)\hat{\phi}_{n+\hat{\mu}} \tag{9}$$

is gauge-invariant. Here, $\hat{\phi}$ is a complex scalar field that transforms under G as

$$\hat{\phi}_n \xrightarrow{G} G(n)\hat{\phi}_n, \qquad \hat{\phi}_n^\dagger \xrightarrow{G} \hat{\phi}_n^\dagger G_n^\dagger\,. \tag{10}$$

b) Follow the steps in the lecture and show that the scalar action

$$S[\hat{\phi},U] = -\sum_{n,\mu>0}\left(\hat{\phi}_n^\dagger U_\mu(n)\hat{\phi}_{n+\hat{\mu}} + \phi_n^\dagger U_\mu^\dagger(n-\hat{\mu})\hat{\phi}_{n-\hat{\mu}} - 2\hat{\phi}_n^\dagger\hat{\phi}_n\right) + \hat{m}^2\sum_n \hat{\phi}_n^\dagger\hat{\phi}_n\,, \tag{11}$$

where $\hat{m}^2 = a^2m^2$, is gauge-invariant.

c) Find the continuum limit of $S[\hat{\phi},U]$.