Quantum Field Theory 2 – Tutorial 1

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Problem 1: One dimensional Gaussian integrals

a) Prove the following identity for $a \in \mathbb{C}$, $\mathrm{Re}(a) > 0$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}. \tag{1}$$

b) Show the generalization

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2+bx} = \sqrt{\frac{2\pi}{a}}e^{\frac{b^2}{2a}}. \tag{2}$$

c) Consider now Gaussian integrals over complex numbers $z = \frac{1}{\sqrt{2}}(x+iy)$ with $\int dzdz^* = \int dxdy$. Show for $w \in \mathbb{C}$, $\mathrm{Re}(w) > 0$ that

$$\int dzdz^*\, e^{-z^*wz} = \frac{2\pi}{w}, \tag{3}$$

and also prove the generalization for arbitrary $u, v \in \mathbb{C}$

$$\int dzdz^*\, e^{-z^*wz+u^*z+z^*v} = \frac{2\pi}{w}e^{u^*w^{-1}v}. \tag{4}$$

Problem 2: Gaussian integrals in more than one dimension

a) Consider a real, symmetric and positive definite N-dimensional matrix $\boldsymbol{A}$ and a real N-component vector $\boldsymbol{x} \in \mathbb{R}^N$, and prove that

$$\int d^N x\, e^{-\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x}} = (2\pi)^{N/2}\frac{1}{\sqrt{\det\boldsymbol{A}}}. \tag{5}$$

Hint: Use an appropriate change of integration variables to diagonalize $\boldsymbol{A}$.

b) Prove the generalization

$$\int d^N x\, e^{-\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x}+\boldsymbol{j}^T\boldsymbol{x}} = (2\pi)^{N/2}\frac{1}{\sqrt{\det\boldsymbol{A}}}\, e^{\frac{1}{2}\boldsymbol{j}^T A^{-1}\boldsymbol{j}}. \tag{6}$$

c) Show for a hermitian and positive matrix $\boldsymbol{A} = \boldsymbol{A}^\dagger$ that

$$\int d^N z\, d^N z^\dagger\, e^{-\boldsymbol{z}^\dagger A\boldsymbol{z}+\boldsymbol{u}^\dagger\boldsymbol{z}+\boldsymbol{z}^\dagger\boldsymbol{v}} = \frac{(2\pi)^N}{\det\boldsymbol{A}}\, e^{\boldsymbol{u}^\dagger A^{-1}\boldsymbol{v}}. \tag{7}$$

Problem 3: Expectation values and Wick's theorem

a) Consider the expectation values with respect to the Gaussian weight

$$\langle \dots \rangle = \frac{\int d^N x \, (\dots)\, e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x}}}{\int d^N x \; e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x}}}. \tag{8}$$

Using Eq. (6) show

$$\begin{aligned} \langle x_m \rangle &= 0, \\ \langle x_m x_n \rangle &= A^{-1}_{mn}, \\ \langle x_m x_n x_p \rangle &= 0, \\ \langle x_m x_n x_p x_q \rangle &= A^{-1}_{mn} A^{-1}_{pq} + A^{-1}_{mp} A^{-1}_{nq} + A^{-1}_{mq} A^{-1}_{np}, \end{aligned} \tag{9}$$

and more general

$$\langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle = \begin{cases} 0 & n \text{ odd} \\ \text{all full contractions} & n \text{ even.} \end{cases} \tag{10}$$

This is a version of Wick's theorem for real bosonic fields.

b) Derive the corresponding relations for complex Gaussian integrals.