Quantum Field Theory 2 – Tutorial 9

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Problem 1: Casimir operator

Consider a Lie algebra with commutation relations

$$[T^a, T^b] = i f^{abc} T^c.$$
⁽¹⁾

Show that the operator $T^2 = T^a T^a$ commutes with all the generators, i.e.,

$$[T^b, T^a T^a] = 0. (2)$$

This implies that T^2 takes a constant value for each irreducible representation t_r^a and the matrix representation is proportional to the unit matrix

$$t_r^a t_r^a = C_2(r) \cdot \mathbb{1} \,. \tag{3}$$

Use the normalization

$$\operatorname{tr}[t_r^a t_r^b] = C(r)\delta^{ab}, \qquad (4)$$

to prove the identity

$$d(r)C_2(r) = d(G)C(r), \qquad (5)$$

where d(r) is the dimension of the (matrix) representation r and d(G) is the dimension of the adjoint representation.