2 Free scalar field

2.1 Classical theory

(1) real scalar field $\phi(x)$ — fundamental field Higgs

composite field, e.g. mesons

pions, ...

more precisely: $\phi(x)$ is a scalar

under Poincaré transformations

(Translations, Rotations, Boosts)

(+ discrete C,P,T)

$$\phi'(x') = \phi(x) \tag{2.1}$$

(2) action should be Lorentz invariant

relevant example: $S[\phi] = \int d^4x \, \mathcal{L}(\phi(x), \partial_\nu \phi(x))$

$$\mathcal{L} = \frac{1}{2} \partial_\nu \phi \, \partial^\nu \phi - V(\phi) \tag{2.2}$$

with $V(\phi) = \frac{1}{2} m^2 \phi^2 + \mathcal{O}(\phi^3)$

constant term: irrelevant at $T=0$

linear term: canceled by shift in ϕ

1+0-dim example see p. 5c

Lorentz trafos: $x_\mu \to \Lambda_\mu{}^\nu x_\nu$ see p. 5a, 5b

with $\Lambda^T \cdot \eta \cdot \Lambda = \eta$

$$\Rightarrow \partial_\mu \phi \to \Lambda_\mu{}^\nu \partial_\nu \phi \tag{2.3}$$

Invariance of $\mathcal{L}$:

$$\partial_\mu \phi \, \partial^\mu \phi \to \partial_\nu \phi \underbrace{\Lambda_\mu{}^\nu \Lambda^\mu{}_\rho}_{(\Lambda^T \eta \Lambda)^\nu{}_\rho} \partial^\rho \phi$$

$$= \partial_\nu \phi \, \partial^\nu \phi \tag{2.4}$$

$$V(\phi) \to V(\phi)$$

(3) Equation of motion (EoM): $\partial_\mu \phi \partial^\mu \phi = (\partial\phi)^2$

$$\delta S = 0 = \delta \int d^4x \left\{ \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} m^2 \phi^2 \right\}$$

$\delta(\partial\phi)^2 = (\delta\, \partial_\mu \phi \, \partial_\nu \phi)\eta^{\mu\nu}$ $\to$ $$= \int d^4x \left\{ \partial_\mu \phi (\partial_\nu \delta\phi) \eta^{\mu\nu} - m^2 \phi \, \delta\phi \right\}$$

partial int. $\to$ $$= - \int d^4x \left\{ \eta^{\mu\nu} \partial_\mu \partial_\nu \phi + m^2 \phi \right\} \delta\phi$$

$$= - \int d^4x \, \delta\phi \, (\partial^2 + m^2) \phi \tag{2.5}$$

$$\Rightarrow \boxed{(\partial^2 + m^2)\, \phi(x) = 0}$$ Klein-Gordon equation (2.6)

Special relativity - basics

Minkowski space:

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \qquad (2.7)$$

$$\underset{\text{contra-variant}}{V^\mu} = \eta^{\mu\nu} \underset{\text{covariant}}{V_\nu}$$

scalar product: $V \cdot W = V^\mu W_\mu$ $\quad (= V^T \eta W)$

$$= V_\mu \eta^{\mu\nu} W_\nu$$

$$= V^\mu \eta_{\mu\nu} W^\nu \qquad (2.8)$$

also

$$\eta^\mu{}_\nu = \eta^{\mu\rho} \eta_{\rho\nu} = \delta^\mu{}_\nu \qquad (2.9)$$

in general

$$T^{\mu_1 \dots \mu_n}{}_{\mu_{n+1} \dots \mu_m} = \eta^{\mu_1 \rho_1} \dots \eta^{\mu_n \rho_n} T_{\mu_1 \dots \mu_m}$$

(2.10)

Poincaré - symmetry :

transformations P that leave

scalar product $(x-y)^2$ invariant:

$$P = (\Lambda, a): \quad x^\mu \xrightarrow{P} x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

with $\Lambda^T \eta \Lambda \overset{!}{=} \eta$

in components $\Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu = \eta_{\mu\nu}$

(2.11)

composition :

$$(\Lambda_1, a_1) \circ (\Lambda_2, a_2) = (\Lambda_1 \circ \Lambda_2, \Lambda_1 a_2 + a_1)$$

exercise: show 2.12

(2.12)

$\Lambda^T \eta \Lambda = \eta$:

$$\Lambda_\sigma{}^\mu \Lambda^\sigma{}_\nu = \eta^\mu{}_\nu = \delta^\mu{}_\nu$$

(2.13)

or $$(\Lambda^{-1})^\mu{}_\sigma = \Lambda_\sigma{}^\mu$$

$\det \Lambda = \pm 1$, $\det \Lambda = 1 \leftarrow SO(1,3)$

1+0 dim theory : (Quantum) Mechanics

from (2.2), p. 4 : $\left.\phi(t,\vec{x})\right|_{1+0\text{-dim}} = \phi(t) = q(t)$

$$\mathcal{L} = \underbrace{\frac{1}{2}\dot{q}^2 - \frac{1}{2}m^2 q^2}_{\text{harmonic oscillator}} - \underbrace{\frac{\lambda}{4} q^4}_{\text{anharmonic term}}$$

EoM : $\partial_t \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$

$$\Rightarrow \quad \ddot{q} + m^2 q + \lambda q^3 = 0$$

(i) $\lambda = 0$: harmonic oscillator

'simply one of the mass points on the oscillating string on p. 1

(ii) 1+d dim : ϕ describes a density of coupled harmonic oscillators

Solution: plane wave $\phi(x) = \varphi_0 \cdot \mathrm{Re}(e^{ikx})$

with $k^2 - m^2 = 0$

$(k^\mu) = (k_0, \vec{k})$ (2.14)

Rest frame: $\vec{k} = 0$

$\Rightarrow k^0 = m$ (or $k^0 = -m$)

particle with mass m

(2.15)

general solution

$$\phi(x) = \int \frac{d^3k}{(2\pi)} \frac{1}{\sqrt{2\omega_k}} \left[\alpha(\vec{k})\, e^{-ikx} + \alpha^*(\vec{k})\, e^{ikx} \right] \quad (2.16)$$

with $k_0 = \sqrt{\vec{k}^2 + m^2} =: \omega_{\vec{k}}$

(i) ϕ is real

(ii) $\partial^2 \phi(x) = -m^2 \phi(x)$

with $\partial^2 e^{\pm ikx}\Big|_{k_0 = \omega_{\vec{k}}} = -k^2 e^{\pm ikx} = -m^2 e^{\pm ikx}$ (2.17)

(iii) $$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + m^2}}\Bigg|_{k_0 = \omega_{\vec{k}}} \simeq \int \frac{d^4k}{(2\pi)^4} (2\pi)\, \delta(k^2 - m^2)$$ Lorentz-invariant measure

(2.18)

(4) complex scalar field

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\underbrace{\phi_1(x) + i\,\phi_2(x)}_{\text{real}}\right) \tag{2.19}$$

action $S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\nu \phi)$

with $\mathcal{L}(\phi, \partial_\nu\phi) = \partial_\nu \phi\, \partial^\nu \phi^* - m^2 \phi\phi^*$

$$= \frac{1}{2}\left[(\partial\phi_1)^2 + (\partial\phi_2)^2 - m^2(\phi_1^2 + \phi_2^2)\right] \tag{2.20}$$

EoM: $(\partial^2 + m^2)\,\phi(x) = 0$

gen. solution:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \Big[\alpha(\vec{k})\, e^{-ikx} + \beta^*(\vec{k})\, e^{ikx}\Big]_{k_0 = \omega_{\vec{k}}} \tag{2.21}$$

Remarks:

(i) S is invariant under $\phi \to e^{i\omega}\phi$: conserved charge
Noether theorem

(ii) Invariance under $\phi \to e^{i\omega(x)}\phi$: gauge symmetry
$\Rightarrow A_\nu(x)$