## 2.2 Noether theorem

'Every continuous symmetry of the action leads to a conserved current density and a conserved charge'

Symmetry transformation (e.g. $\delta_\varepsilon \phi = i\varepsilon\phi$
$\phi \to e^{i\alpha}\phi$ )

$$\phi(x) \to \phi(x) + \delta_\varepsilon \phi(x) \tag{2.22}$$

with $S \to S[\phi(x) + \delta_\varepsilon \phi(x)]\Big|_{EoM} \overset{!}{=} S[\phi(x)]$

This is satisfied for

$$\mathcal{L} \to \mathcal{L} + \varepsilon\, \partial_\nu J^\nu(\phi) \tag{2.23}$$

total derivative

$$\Rightarrow \int d^4x\, \mathcal{L} \to \int d^4x\, \mathcal{L} + \varepsilon \underbrace{\int d^4x\, \partial_\nu J^\nu(\phi)}_{0 \text{ if } J(\phi)\big|_{\text{infinity}} = 0} \tag{2.24}$$

It follows that

$$\mathcal{L} \to \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\varepsilon \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\mu \delta_\varepsilon \phi$$

$$= \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\varepsilon \phi + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta_\varepsilon \phi \right)$$

$$- \left( \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \delta_\varepsilon \phi$$

Euler-Lagrange

$$= \mathcal{L} + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta_\varepsilon \phi \right) + (\mathrm{EoM})\, \delta_\varepsilon \phi$$

$$\overset{!}{=} \mathcal{L} + \varepsilon\, \partial_\mu J^\mu \qquad (2.25)$$

Evaluated at the EoM: $\left. \frac{\partial \delta_\varepsilon \phi}{\partial \varepsilon} \right|_{\varepsilon=0} = \Delta \phi$

$$\partial_\mu J^\mu = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \Delta \phi \right) \qquad (2.26)$$

or

$$\boxed{j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \Delta \phi - J^\mu}$$ conserved current

Noether Charge:

$$\boxed{Q(t) = \int d^3x \; j^0(t,\vec{x})}$$

$$\dot{Q}(t) = \int d^3x \; \partial_0 j^0(t,\vec{x})$$

$$= + \int d^3x \; \vec{\partial} \, \vec{j}(t,\vec{x}) = 0 \qquad (2.27)$$

Remarks

(i) Extends to general symmetry with $r$ parameters: $\Delta_r \phi = \left.\frac{\partial \delta_\varepsilon \phi}{\partial \varepsilon_r}\right|_{\varepsilon=0}$

$$\boxed{j^\nu_r = \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_i} \Delta_r \phi_i - \mathcal{J}^\nu_r} \qquad (2.28)$$

multiple fields

(ii)* Split of $\delta_\varepsilon \phi = \delta_{\phi_\varepsilon} \phi + \delta_\varepsilon x_\nu \partial^\nu \phi$

Trafo of $\phi$     Trafo of $x$

leads to, $\Delta x = \frac{\partial \delta_\varepsilon x}{\partial \varepsilon_r}$

$$j^\nu_r = \Delta_r \phi \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} + \Delta_r x^\nu \left\{ \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} - \mathcal{L} \eta^\mu{}_\nu \right\} - \mathcal{J}^\nu_r \qquad (2.29)$$

(1) Energy-momentum conservation

Symmetry: Translations

'Physics is invariant under a shift of the lab'

$$\phi(x) \to \phi(x+a) \qquad (2.30)$$

Infinitesimally

$$\phi(x) \to \phi(x+\varepsilon) = \phi(x) + \varepsilon^\nu \overbrace{\partial_\nu \phi(x)}^{\Delta_\nu \phi} + \mathcal{O}(\varepsilon^2) \qquad (2.31)$$

It follows that

$$\mathcal{L}(\phi(x), \partial_\nu \phi(x)) \to \mathcal{L} + \varepsilon^\nu \partial_\nu \mathcal{L} + \mathcal{O}(\varepsilon^2)$$
$$= \mathcal{L} + \varepsilon^\nu \partial_\nu \eta^\nu{}_\nu \mathcal{L} + \mathcal{O}(\varepsilon^2) \qquad (2.32)$$

With $r=\nu$ we have $\mathcal{L} \to \mathcal{L} + \varepsilon^r \partial_\nu J^\nu_r$ with

$$J^\nu{}_r = \eta^\nu{}_r \mathcal{L} \qquad (2.33)$$

Noether current (2.28):

$$\boxed{j^{\mu}{}_{\nu} = \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \eta^{\mu}{}_{\nu} \mathcal{L} =: T^{\mu}{}_{\nu}} \qquad (2.34)$$

$T^{\mu\nu}$: energy-momentum tensor

$$\partial_\mu T^{\mu\nu} = 0 \qquad (2.35)$$

- four conserved currents / charges

$$p^{\mu} = \int d^3x \, T^{0\mu} \qquad (2.36)$$

4-momentum

energy(-density)

$$p^0 = \int d^3x \, T^{00} = \int d^3x \underbrace{\Big[\partial^0 \phi \overbrace{\frac{\partial \mathcal{L}}{\partial \partial_0 \phi}}^{\pi} - \mathcal{L}\Big]}_{}$$

$$= H = \int d^3x \, \mathcal{H} \qquad \text{Hamiltonian} \qquad (2.37)$$

Scalar field: $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V$

$$\mathcal{H} = (\partial_0\phi)^2 - \mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)$$

Remarks

- Covariance of $P^\nu$ is not apparent

Note, however, that

$$\int d^3x \underbrace{\frac{\partial \mathcal{L}}{\partial \partial_0 \phi}}_{\text{0-comp. of four-vector}} \sim d^3x \, dx^0 \qquad (2.38)$$

$d^4x$ is invariant! Exercise: Show covariance of $P^\nu$.

- $P^i$ generates translations:

$$P^i = \int d^3x \, T^{0i} = \int d^3x \overbrace{\frac{\partial \mathcal{L}}{\partial \partial_0 \phi}}^{\pi} \partial^i \phi$$

real scalar field (2.2), p. 4

$$= \int d^3x \, \pi \, \vec{\nabla} \phi \qquad (2.39)$$

Poisson bracket: $\{P^i, \phi(\vec{x})\} = -\vec{\nabla}\phi(\vec{x})$

with $\{\phi(\vec{x}), \pi(\vec{y})\} = \delta(\vec{x} - \vec{y})$ (2.40)

Quantisation:

$$\{A, B\} \to [\hat{A}, \hat{B}] \qquad (2.41)$$

- in general, $T^{\mu\nu}$ in (2.34) is <u>not</u> symmetric due to $\partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}$, but always can be symmetrised. The latter property is important for the coupling to gravity.

Alternatively, one can define

$$\boxed{T^{\mu\nu}_{sym} = \frac{1}{\sqrt{-\det g}} \frac{\delta S}{\delta g^{\mu\nu}}} \qquad (2.42)$$

[with $\frac{\delta g^{\alpha\beta}(x)}{\delta g^{\mu\nu}(y)} = \frac{1}{2}\left(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu\right)$, $\frac{\delta \sqrt{-g(x)}}{\delta g^{\mu\nu}(y)} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta(x-y)$]

- * $\phi'(x') = \phi(x)$ for scalar field

(So far we have used $\phi(x) \to \phi(x')$)

$$\Rightarrow \Delta\phi = 0$$

$$\Delta_\rho x^\nu = \eta_\rho{}^\nu$$

$$J^\mu{}_\rho = 0 \qquad (\mathcal{L}' = \mathcal{L})$$

Check that $j^\mu{}_\rho$ in (2.29), p.10 is (2.39)

(2) Charge of a complex scalar field

Action (2.20), p. 7: $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$\mathcal{L} = \partial_\nu \phi \, \partial^\nu \phi^* - m^2 \phi \phi^*$$

U(1)-Symmetry:

$$\phi \to \phi' = e^{i\varepsilon} \phi = \phi + i\varepsilon\phi + O(\varepsilon^2)$$

$$\sim \phi^* \to \phi^{*\prime} = \phi^* e^{-i\varepsilon} = \phi - i\varepsilon\phi + O(\varepsilon^2)$$

(2.43)

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} \Rightarrow J^\nu = 0$$

Noether current (2.28), p. 10, $\Delta\phi = i\phi$
$\Delta\phi^* = -i\phi$

$$j^\nu = \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \Delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi^*} \Delta\phi^*$$

$$= \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} i\phi - \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi^*} i\phi^* \qquad (2.44)$$

$$= i\left[\partial^\nu \phi^* \, \phi - \partial^\nu \phi \, \phi^*\right]$$

Noether charge (2.27), p. 10

$$Q = \int d^3x \, j^0 = i \int d^3x \left[ \phi^* \partial_t \phi - \partial_t \phi^* \phi \right] \quad (2.45)$$

Conservation law:

$$\dot{Q}\Big|_{EoM} = i \int d^3x \Big[ \cancel{\dot{\phi}^* \dot{\phi} - \dot{\phi}^* \dot{\phi}}$$
$$+ \phi^* \partial_t^2 \phi - \partial_t \phi^* \phi \Big]$$

$$\text{EoM} \rightarrow \quad = i \int d^3x \Big[ \phi^* (\Delta - m^2) \phi - (\Delta - m^2) \phi^* \phi \Big] \qquad \Delta = \vec{\nabla}^2$$

$$= i \int d^3x \left[ \phi^* \Delta \phi - (\Delta \phi^*) \phi \right] = 0$$

part. int.

(2.46)

In momentum space (with (2.21), p. 7)

$$Q = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \Big\{ \underbrace{\alpha^*(\vec{p}) \, \alpha(\vec{p})}_{\text{part.}} - \underbrace{\beta^*(\vec{p}) \, \beta(\vec{p})}_{\text{anti-part.}} \Big\}$$

Normalisation of $\alpha, \beta$

Quantisation: $\alpha, \beta \rightarrow$ op. $a, b$

(2.47)