2.3 Quantisation

(1) QFT is the field-theoretical limit of quantum mechanics. Therefore

$$[\hat{q}, \hat{p}] = i\hbar \qquad\qquad [\hat{\phi}(\vec{x}), \hat{u}(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$$

$$[\hat{q},\hat{q}] = 0 = [\hat{p},\hat{p}] \longrightarrow [\hat{\phi}(\vec{x}),\hat{\phi}(\vec{y})] = 0 = [\hat{\Pi}(\vec{x}),\hat{u}(\vec{y})]$$

$$(2.48)$$

0+1 dim QFT $\simeq$ QM

Example: harmonic oscillator

see page 17a

Quantisation: • $\phi(x) \longrightarrow \hat{\phi}(x)$ operator

• expectation value $\langle \hat{\phi}(x) \rangle$

$\simeq$ classical field

Canonical momentum:

$$(2.49)$$

$$\boxed{\Pi(t,\vec{x}) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi}(t,\vec{x}) = \partial^0 \phi(t,\vec{x})}$$

with $\mathcal{L} = \frac{1}{2}\{\partial_0\phi\partial^0\phi - (\vec{\nabla}\phi)^2 - m^2\phi^2\}$

free real scalar field

Example: 1 + 0 dimensions

$$S[q] = \int dt \left\{ \underbrace{\frac{1}{2} \dot{q}^2}_{(\partial_t \phi)^2} - \underbrace{\frac{1}{2} \omega^2 q^2}_{\phi(-\vec{\partial}^2 + m^2)\phi} \right\}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{L}$$

$$H = p \cdot \dot{q} - L$$

with $\quad p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$

$$\Rightarrow \quad H = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2$$

Quantisation: $\quad p, q \rightarrow \hat{p}, \hat{q}$

$$\text{with} \quad [\hat{q}, \hat{p}] = i$$

creation, annihilation ops.: $\boxed{\text{from now on: Drop "} \wedge \text{"}}$

$$q = \frac{1}{\sqrt{2\omega}} (a + a^+)$$

$$p = -i\sqrt{\frac{\omega}{2}} (a - a^+)$$

with $\quad [a, a^+] = 1$

$$\Rightarrow \boxed{\hat{H} = \left(a^+ a + \frac{1}{2}\right)\omega}$$

t - dependence:

Heisenberg picture: $i \frac{\partial}{\partial t} O(t) = [O(t), H]$

$$\Rightarrow O(t) = e^{iHt} \overbrace{O(0)}^{O} e^{-iHt}$$

i.e.: $H(t) = H(0) = H$

It follows that

$$[q(t), p(t)] = i$$

$$\text{``} e^{iHt} \underbrace{[q, p]}_{i} {}^{//} e^{-iHt}$$

Hamiltonian density: $\Delta = \vec{\partial}^2$, _real scalar_

$$\mathcal{H} = \pi \cdot \partial_0 \phi - \mathcal{L} \qquad \text{partial int.} \qquad (2.50)$$

$$= \frac{1}{2} \left\{ \pi(t,\vec{x})^2 + \phi(t,\vec{x})(-\Delta + m^2)\phi(t,\vec{x}) \right\}$$

Canonical commutation relations: $\pi = \partial^0 \phi = \dot{\phi}$

$$\boxed{[\phi(t,\vec{x}), \dot{\phi}(t,\vec{y})] = i\delta^3(\vec{x} - \vec{y})}$$

$$(2.51)$$

and $[\phi(t,\vec{x}), \phi(t,\vec{y})] = 0$

Remarks:

(i) ϕ satisfies EoM as does $\langle \phi \rangle$
op. expect. value

(ii) The free field theory describes

coupled set of harm. osc. due to $\underline{\phi \Delta \phi}$

In Fourier space: $\phi \Delta \phi \to \phi(-p)\,\vec{p}^2\,\phi(p)$

$$\Rightarrow \mathcal{L}, \mathcal{H} \text{ diagonal}$$

$\rightsquigarrow$ Introduction of creation/annihilation

ops. $a^+(\vec{p})$, $a(\vec{p})$ respectively

(2) Fourier space:

$\phi(x)$ is defined as ϕ in eq. (2.16), p. 6

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a(\vec{p})\, e^{-ipx} + a^+(\vec{p})\, e^{ipx} \right]_{p_0 = \omega_{\vec{p}}}$$

operators

(2.52a)

with $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

$$\pi(x) = \partial^0 \phi(x) = -i \int \frac{d^3 p}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{p}}}{2}} \left[a(\vec{p})\, e^{-ipx} - a^+(\vec{p})\, e^{ipx} \right]_{p_0 = \omega_{\vec{p}}}$$

(2.52b)

Fourier transforms:

$$\tilde{\phi}(p) := \int d^4x\, e^{ipx} \phi(x)$$

(2.53a)

$$\phi(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \tilde{\phi}(p)$$

Spatial Fourier transform $(t=0)$ see p. 17a-b)

$$\tilde{\phi}(\vec{p}) := \int d^3x\, e^{-i\vec{p}\vec{x}} \phi(\vec{x})$$

(2.53b)

$$= \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a(\vec{p}) + a^+(-\vec{p}) \right]$$

(a) Canonical commutation relations:

$$[\tilde{\phi}(\vec{p}), \tilde{\pi}(\vec{q})] = \int d^3x\, d^3y\, e^{-i(\vec{p}\vec{x}+\vec{q}\vec{y})} \underbrace{[\phi(\vec{x}), \pi(\vec{y})]}_{i\delta^3(\vec{x}-\vec{y})}$$

$$= i\int d^3x\, e^{i(\vec{p}+\vec{q})\vec{x}} = i(2\pi)^3 \delta^3(\vec{p}+\vec{q})$$

$$(2.54)$$

Trivially:
$$[\tilde{\phi}(\vec{p}), \tilde{\phi}(\vec{q})] = 0 = [\tilde{\pi}(\vec{p}), \tilde{\phi}(\vec{q})]$$

That is, $\tilde{\pi}(\vec{q})$ is conjugate to $\tilde{\phi}(-\vec{q})$.

Commutation relations for $a, a^\dagger$:

Parameterise $a, a^\dagger$ in $\tilde{\phi}, \tilde{\pi}$:

$(2.52b) \longrightarrow$
$$\boxed{\tilde{\pi}(\vec{p}) = \int d^3x\, e^{-i\vec{q}\vec{x}} \partial^0\phi(\vec{x}) = -i\sqrt{\frac{\omega_{\vec{p}}}{2}}\left\{a(\vec{p}) - a^\dagger(-\vec{p})\right\}}$$

$$(2.55)$$

From $(2.53b)$, (2.55):

$$\boxed{\begin{aligned} a(\vec{p}) &= \sqrt{\frac{\omega_{\vec{p}}}{2}}\,\tilde{\phi}(\vec{p}) + i\sqrt{\frac{1}{2\omega_{\vec{p}}}}\,\tilde{\pi}(\vec{p}) \\ a^\dagger(-\vec{p}) &= \sqrt{\frac{\omega_{\vec{p}}}{2}}\,\tilde{\phi}(\vec{p}) - i\sqrt{\frac{1}{2\omega_{\vec{p}}}}\,\tilde{\pi}(\vec{p}) \end{aligned}}$$

$$(2.56)$$

Analogous to harmonic oscillator in QM,

Commutator $[a, a^\dagger]$:

$$\left[a(\vec{p}), a^\dagger(\vec{p}) \right] = -\frac{i}{2} \sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}} \left[\tilde{\phi}(\vec{p}), \hat{\tilde{\pi}}(-\vec{q}) \right]$$

$$-\frac{i}{2} \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \left[\tilde{\phi}(-\vec{q}), \hat{\tilde{\pi}}(\vec{p}) \right]$$

$$= (2\pi)^3 \, \delta(\vec{p} - \vec{q}) \qquad (2.57)$$

also $\left[a(\vec{p}), a(\vec{q}) \right] = \left[a^\dagger(\vec{p}), a^\dagger(\vec{q}) \right] = 0 \qquad (2.58)$

The creation & annihilation operators have canonical commutation relations of a density of harmonic oscillators.

(b) Hamiltonian density diagonal in momentum space:

$$\Delta e^{-i\vec{q}\vec{x}} = -\vec{q}^2 e^{-i\vec{q}\vec{x}}$$

$$\downarrow$$

$$-\int d^3x \; \phi(\vec{x}) \Delta \phi(\vec{x}) = \int d^3x \, \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} e^{-i\vec{x}(\vec{p}+\vec{q})} \tilde{\phi}(\vec{p}) \, \vec{q}^2 \, \tilde{\phi}(\vec{q})$$

$$= \int \frac{d^3p}{(2\pi)^3} \; \tilde{\phi}(\vec{p}) \, \vec{p}^2 \, \tilde{\phi}(-\vec{p})$$

$$(2.59)$$

Analogously

$$m^2 \int d^3x \; \phi^2(\vec{x}) = m^2 \int \frac{d^3p}{(2\pi)^3} \tilde{\phi}(\vec{p}) \, \tilde{\phi}(-\vec{p})$$

$$\int d^3x \; \pi^2(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \tilde{\pi}(\vec{p}) \, \tilde{\pi}(-\vec{p})$$

$$(2.60)$$

$\Rightarrow$ Diagonal Hamiltonian

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left\{ \tilde{\pi}(\vec{p}) \tilde{\pi}(-\vec{p}) + \omega_p^2 \, \tilde{\phi}(\vec{p}) \tilde{\phi}(-\vec{p}) \right\}}$$

$$(2.61)$$

with

$$\omega_{\vec{p}}^2 = \vec{p}^2 + m^2$$

Physics interpretation of Hamiltonian
H is best done in terms of $a, a^\dagger$:

Use
$$\tilde{\phi}(\vec{p}) = \frac{1}{\sqrt{2\omega_{\vec{p}}}}\left[a(\vec{p}) + a^\dagger(-\vec{p})\right]$$
(2.536)

$$\tilde{\pi}(\vec{p}) = -i\sqrt{\frac{\omega_{\vec{p}}}{2}}\left[a(\vec{p}) - a^\dagger(-\vec{p})\right]$$
(2.55)

As in the QM-example (p. 17 a-b) it follows

$$H = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\,\omega_{\vec{p}}\left\{a^\dagger(\vec{p})a(\vec{p}) + \frac{1}{2}\left(a^\dagger(\vec{p})a^\dagger(-\vec{p}) + a(\vec{p})a(-\vec{p})\right)\right.$$

$$+ a^\dagger(\vec{p})a(\vec{p}) - \frac{1}{2}\left(a^\dagger(\vec{p})a^\dagger(-p) + a(\vec{p})a(-\vec{p})\right)$$

$$+ \underbrace{\left[a(\vec{p}), a^\dagger(\vec{p})\right]}_{(2\pi)^3\delta(0)}\left.\vphantom{\frac{1}{2}}\right\}\quad\leftarrow \text{eq. (2.57)}$$

$$= \int\frac{d^3p}{(2\pi)^3}\,\omega_{\vec{p}}\,a^\dagger(\vec{p})a(\vec{p}) + \frac{1}{2}V\int\frac{d^3p}{(2\pi)^3}\,\omega_{\vec{p}}$$

(2.62)

with

$$(2\pi)^3\delta^3(0) = \int d^3x\, e^{i\vec{x}(\vec{p}-\vec{p})} = V$$

Volume of $\mathbb{R}^3$

Remarks:

(i) In QFT the vacuum energy density is divergent: $\frac{1}{2} \int \frac{d^3 P}{(2\pi)^3} \, \omega_{\vec{P}}$

(ii) For QFT in a finite Volume the volume factor is finite, e.g. $\mathbb{R}^3 \to T^3, S^3$

(iii) The infinite constant will be dropped from now on; it does not play a rôle for energy differences.

It does, however, play a rôle

- at finite Temperature
- QFT coupled to gravity

  'cosmological constant problem'

- QFT with boundary conditions

  Casimir effect in QED:

  attractive force between conducting plates

Finally, we have the Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} \, \omega_{\vec{p}} \, a^\dagger(\vec{p}) a(\vec{p})$$

(2.63)

with $\quad \omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$, $\quad [\, a(\vec{p}), a^\dagger(\vec{q}) \,] = (2\pi)^3 \delta(\vec{p} - \vec{q})$

H is the Hamiltonian of a momentum continuum of harmonic oscillators with frequencies $\omega_{\vec{p}}$. Interpretation of $a, a^\dagger$ is that of annihilation and creation operators respectively.

(c) Hilbert space construction (Fock space)

(i) Vacuum (ground state) $|0\rangle$:

$$H|0\rangle = 0$$

with $\quad \boxed{a(\vec{p})|0\rangle = 0}$ (2.64)

Normalised : $\langle 0|0\rangle = 1$

All states are generated by applying $a, a^\dagger$ on the vacuum $|0\rangle$:

$a, a^\dagger$ are annihilation and creation ops.

(ii) One-particle state: $\boxed{H|\vec{p}\rangle = \omega_{\vec{p}}|\vec{p}\rangle}$ p26a

$$\boxed{|\vec{p}\rangle = \sqrt{2\omega_{\vec{p}}}\, a^\dagger(\vec{p})|0\rangle} \qquad (2.65)$$

Orthogonality: $\langle\vec{p}| = |\vec{p}\rangle^\dagger = \langle 0|a(\vec{p})$

$$\langle\vec{p}|\vec{p}'\rangle = 2\sqrt{\omega_{\vec{p}}\,\omega_{\vec{p}'}}\,\langle 0|\,a(\vec{p}')\,a^\dagger(\vec{p})|0\rangle$$

$$a|0\rangle = 0 \rightarrow = 2\sqrt{\omega_{\vec{p}}\,\omega_{\vec{p}'}}\,\langle 0|\,[\,a(\vec{p}'),\,a^\dagger(\vec{p})\,]\,|0\rangle$$

$$= (2\pi)^3\, 2\omega_{\vec{p}}\,\delta(\vec{p}-\vec{p}') \qquad (2.66)$$

General one-particle state:

$$\boxed{|f\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\,f(\vec{p})\,a^\dagger(\vec{p})|0\rangle} \qquad (2.67)$$

$$\Rightarrow \langle f|f\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\,f(p)\,f^*(p)$$

properties under Lorentz trafo's $\rightarrow$ (vi)

$|\vec{p}\rangle$ is eigenstate of H_0:

$$H\,|\,\vec{p}\rangle = \int \frac{d^3p'}{(2\pi)^3}\,\omega_{\vec{p}'}\,a^\dagger(\vec{p}')\,a(\vec{p}')\,a^\dagger(\vec{p})\,|0\rangle\,\sqrt{2\omega_{\vec{p}}}$$

$$= \int \frac{d^3p'}{(2\pi)^3}\,\omega_{\vec{p}'}\,a^\dagger(\vec{p}')\underbrace{\left[a(\vec{p}'),\,a^\dagger(\vec{p})\right]}_{(2\pi)^3\,\delta(\vec{p}-\vec{p}')}|0\rangle\,\sqrt{2\omega_{\vec{p}}}$$

$$= \omega_{\vec{p}}\,a^\dagger(\vec{p})\,|0\rangle \cdot \sqrt{2\omega_{\vec{p}}}$$

$$= \omega_{\vec{p}}\,|\,\vec{p}\rangle$$

For general one-particle state:

$$H\,|\,f\rangle = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\vec{p}}^3}{2}}\,f(\vec{p})\,a^\dagger(\vec{p})\,|0\rangle$$

N-particle state:

$$H\,|\,\vec{p}_n \ldots, \vec{p}_1\rangle$$

$$= \left(\sum_{i=1}^{n}\omega_{\vec{p}_i}\right)|\,\vec{p}_n \ldots \vec{p}_1\rangle$$

(iii) N-particle states:
$$\boxed{H|\vec{p}_n \cdots \vec{p}_1\rangle = \left(\sum_{i=1}^{n} \omega_{\vec{p}_i}\right)|\vec{p}_n \cdots \vec{p}_1\rangle}$$

$$\sqrt{2\omega_{\vec{p}_2} 2\omega_{\vec{p}_1}}\, a^\dagger(\vec{p}_2)\, a^\dagger(\vec{p}_1)|0\rangle = |\vec{p}_2\, \vec{p}_1\rangle$$

$$\sqrt{\cdots}\, a^\dagger(\vec{p}_3)\, a^\dagger(\vec{p}_2)\, a^\dagger(\vec{p}_1)|0\rangle = |\vec{p}_3\, \vec{p}_2\, \vec{p}_1\rangle$$

$$\vdots$$

$$\text{(2.68)}$$

- states have Bose symmetry:

$$|\vec{p}_n, \cdots \vec{p}_{i+1}\, \vec{p}_i \cdots \vec{p}_1\rangle = |\vec{p}_n, \cdots \vec{p}_i\, \vec{p}_{i+1}, \cdots \vec{p}_1\rangle$$

$$\text{(2.69)}$$

as

$$a^\dagger(\vec{p}_{i+1})\, a^\dagger(\vec{p}_i) = a^\dagger(\vec{p}_i)\, a^\dagger(\vec{p}_{i+1})$$

- energy-momentum is additive:

take some state $|\beta\rangle$, then

$a^\dagger(\vec{p})|\beta\rangle$ is a state with <u>one</u>

additional particle with momentum $\vec{p}$:

Sketch: $H|\beta\rangle = E_\beta |\beta\rangle$

then
$$H\left(a^\dagger(\vec{p})|\beta\rangle\right) = a^\dagger(\vec{p})\, H|\beta\rangle + \overbrace{[H, a^\dagger(\vec{p})]}^{\omega_{\vec{p}} a^\dagger(\vec{p})}|\beta\rangle$$

$$= \left(H + \omega_{\vec{p}}\right)\left(a^\dagger(\vec{p})|\beta\rangle\right)$$

$$\text{(2.70)}$$

- annihilation:

  $|a(\vec{p})|\beta\rangle$ is a state, where

  a particle with momentum $\vec{p}$

  is removed.

  Example: consider one-particle state $|f\rangle$

  (2.67), p. 26

$$a(\vec{p})|f\rangle = a(\vec{p}) \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}'}}} f(\vec{p}') a^\dagger(\vec{p}')|0\rangle$$

$$= \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}'}}} f(\vec{p}') [a(\vec{p}), a^\dagger(\vec{p}')]|0\rangle$$

$$= \frac{1}{\sqrt{2\omega_{\vec{p}}}} f(\vec{p})|0\rangle \qquad\qquad (2.71)$$

(iv) Interpretation of $\phi(x)$

- states with defined particle number

  n have vanishing expectation value

  of ϕ, as ϕ creates and annihilates

  a particle.

For example: $\langle 0| \phi(x) | 0 \rangle = 0$

with $\langle 0| a | 0 \rangle = 0 = \langle 0| a^\dagger | 0 \rangle$

also $\langle \vec{p} | \phi(x) | \vec{p} \rangle = 0$

$$\vdots$$

$$\langle \vec{p}_1 \cdots \vec{p}_n | \phi(x) | \vec{p}_n \cdots \vec{p}_1 \rangle = 0$$

with $|\vec{p}_n \cdots \vec{p}_1 \rangle \simeq a^\dagger(\vec{p}_n) \cdots a^\dagger(\vec{p}_1) | 0 \rangle$

$$(2.72)$$

Classical field:

$$\langle \alpha | \phi(x) | \alpha \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left\{ e^{-ipx} \alpha(\vec{p}) + e^{ipx} \alpha^*(\vec{p}) \right\}$$

$$(2.73)$$

with coherent state $|\alpha\rangle$:

$$a(\vec{p}) | \alpha \rangle = \alpha(\vec{p}) | \alpha \rangle \qquad (2.74)$$

unchanged by annihilation (detection)

of a particle with momentum p.

It follows that

$$|\alpha\rangle = \frac{1}{N(\alpha)} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{\infty} \left[\int \frac{d^3p_i}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}_i}}} \alpha(\vec{p}_i) \right] |\vec{p}_1, \ldots, \vec{p}_n\rangle \tag{2.75}$$

with

$$a(\vec{p}) |\vec{p}_1 \cdots \vec{p}_n\rangle = \sum_{i=1}^{n} (2\pi)^3 |\vec{p}_1 \cdots \vec{p}_{i-1} \vec{p}_{i+1} \cdots \vec{p}_n\rangle$$
$$\sqrt{2\omega_{\vec{p}_i}} \, \delta(\vec{p} - \vec{p}_i) \tag{2.76}$$

Normalisation $N(\alpha)$ such that

$$\langle \alpha | \alpha \rangle = 1 \tag{2.77}$$

$$\Rightarrow \quad N(\alpha) = \exp\left\{ \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} |\alpha(\vec{p})|^2 \right.$$

and finally

$$\boxed{ |\alpha\rangle = \frac{1}{N(\alpha)} \exp\left\{ \int \frac{d^3p}{(2\pi)^3} \alpha(\vec{p}) a^\dagger(\vec{p}) \right\} |0\rangle }$$
$$\tag{2.78}$$

Some technical details:

From (2.76) it follows that

$$\frac{1}{n!} a(\vec{p}) \left[\int \frac{d^3 p'}{(2\pi)^3} \alpha(\vec{p'}) a^\dagger(\vec{p'}) \right]^n |0\rangle$$

$$= \frac{1}{n!} n \, \alpha(\vec{p}) \left[\int \frac{d^3 p'}{(2\pi)^3} \alpha(\vec{p'}) a^\dagger(\vec{p'}) \right]^{n-1} |0\rangle$$

$$= \alpha(\vec{p}) \frac{1}{(n-1)!} \left[\int \frac{d^3 p'}{(2\pi)^3} \alpha(\vec{p'}) a^\dagger(\vec{p'}) \right]^{n-1} |0\rangle$$

and similarly

$$\langle 0| \frac{1}{n!} \left[\int \frac{d^3 p'}{(2\pi)^3} \alpha^*(\vec{p'}) a(\vec{p'}) \right]^n a^\dagger(\vec{p})$$

$$= \langle 0| \frac{1}{(n-1)!} \left[\int \frac{d^3 p'}{(2\pi)^3} \alpha^*(\vec{p'}) a(\vec{p'}) \right]^{n-1} \alpha^*(\vec{p})$$

with $\quad a(\vec{p}) \, a^\dagger(\vec{p_1}) \cdots a^\dagger(\vec{p_n}) |0\rangle$

$$= \left(\underbrace{\left[a(\vec{p}), a^\dagger(\vec{p_1}) \right]}_{(2\pi)^3 \delta(\vec{p} - \vec{p_1})} + a^\dagger(\vec{p_1}) \cdot a(\vec{p}) \right) a^\dagger(\vec{p_2}) \cdots a^\dagger(\vec{p_n}) |0\rangle$$

$$= (2\pi)^3 \, \delta(\vec{p} - \vec{p_1}) \, a^\dagger(\vec{p}) \, a^\dagger(\vec{p_2}) \cdots a^\dagger(\vec{p_n}) |0\rangle$$
$$+ a^\dagger(\vec{p_1}) a(\vec{p}) \, a^\dagger(\vec{p_2}) \cdots a^\dagger(\vec{p_n}) |0\rangle$$

$$\omega_i = \omega(\vec{k}_i) = \sqrt{\vec{k}_i^2 + m^2} \quad , \quad \omega_{i'} = \omega(\vec{k}_i')$$

$$\langle \alpha | \alpha \rangle = \frac{1}{N^2(\alpha)} \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right)^2 \int \prod_{i=1}^{n} \left[\frac{d^3 k_i}{(2\pi)^3} \frac{d^3 k_i'}{(2\pi)^3} \cdot \alpha^*(\vec{k}_i) \alpha(\vec{k}_i') \right]$$

$$\cdot \langle 0| a(\vec{k}_1) \circ\circ\circ a(\vec{k}_n) a^\dagger(\vec{k}_n') \circ\circ\circ a(\vec{k}_1') |0\rangle$$

$$\langle 0| a(\vec{k}_1) \circ\circ\circ \underbrace{a(\vec{k}_n) \cdot a^\dagger(\vec{k}_n')}_{\underbrace{[a(\vec{k}_n), a^\dagger(\vec{k}_n')]}_{(2\pi)^3 \delta(\vec{k}_n - \vec{k}_n')} + a^\dagger(\vec{k}_n') a(\vec{k}_n)} \circ\circ\circ a(\vec{k}_1') |0\rangle$$

$$= (2\pi)^3 \delta(\vec{k}_n - \vec{k}_n') \langle 0| a(\vec{k}_1) \circ\circ\circ a(\vec{k}_{n-1}) \cdot a^\dagger(\vec{k}_{n-1}') \circ\circ\circ a^\dagger(\vec{k}_1') |0\rangle$$

$$+ \langle 0| a(\vec{k}_1) \circ\circ\circ a(\vec{k}_{n-1}) a^\dagger(\vec{k}_n') \underbrace{a(\vec{k}_n) a^\dagger(\vec{k}_{n-1}')}_{\underbrace{[a(\vec{k}_n), a^\dagger(\vec{k}_{n-1}')]}_{(2\pi)^3 \delta(\vec{k}_n - \vec{k}_{n-1}')} + a^\dagger(\vec{k}_{n-1}')} \circ\circ\circ a^\dagger(\vec{k}_1') |0\rangle$$

$$\cdot a(\vec{k}_n)$$

$$\vdots$$

$$= (2\pi)^3 \sum_{i=1}^{n} \langle 0| a(\vec{k}_1) \circ\circ\circ a(\vec{k}_{n-1}) a^\dagger(\vec{k}_n') \circ\circ\circ \widehat{a^\dagger(\vec{k}_i')} \circ\circ\circ a^\dagger(\vec{k}_1') |0\rangle$$

$$\cdot \delta(\vec{k}_n - \vec{k}_i')$$

where $\widehat{a^\dagger(\vec{k}_i')} = 1$

$$\langle \alpha | \alpha \rangle = \frac{1}{N^2(\alpha)} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(n-1)!} \int \frac{d^3 k_n}{(2\pi)^3} \alpha^*(\vec{k}_n) \alpha(k_n)$$

$$\cdot \int \prod_{i=1}^{n-1} \left[\frac{d^3 k_i}{(2\pi)^3} \frac{d^3 k_i'}{(2\pi)^3} \alpha^*(\vec{k}_i) \alpha(\vec{k}_i') \right] \langle 0 | a(\vec{k}_1) \cdots a(\vec{k}_{n-1}) a^\dagger(k_{n-1}')$$

$$\cdots a(\vec{k}_1) | 0 \rangle$$

$$\circ$$
$$\circ$$
$$\circ$$

$$= \frac{1}{N^2(\alpha)} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\int \frac{d^3 k}{(2\pi)^3} \alpha^*(\vec{k}) \alpha(\vec{k}) \right]^n$$

$$= \frac{1}{N^2(\alpha)} e^{\int \frac{d^3 k}{(2\pi)^3} |\alpha(\vec{k})|^2}$$

$$\Rightarrow \boxed{N(\alpha) = e^{\frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} |\alpha(\vec{k})|^2}}$$

Scalar product :

$$\langle \alpha' | \alpha \rangle = \exp \left\{ -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \overbrace{\left(|\alpha(\vec{p})|^2 + |\alpha'(\vec{p})|^2 - 2\alpha'^* \alpha(\vec{p}) \right)}^{1/N(\alpha)N(\alpha')} \right\}$$

$$(2.79)$$

not orthogonal.

Completeness : (1+0 dim)

$$\frac{1}{\pi} \int d^2\alpha \, |\alpha\rangle\langle\alpha| = \mathbb{1} \qquad (2.80)$$

(v)· Conserved energy – momentum tensor $\partial_\nu T^{\mu\nu} = 0$

$$P^0 = H$$

$$\vec{P} = \int d^3x \, \pi \, \vec{\nabla} \phi$$

$$\approx \int \frac{d^3p}{(2\pi)^3} \, \vec{p} \, a^+(\vec{p}) a(\vec{p})$$

$$(2.81)$$

with

$$\vec{P} | \vec{p} \rangle = \vec{p} \, | \vec{p} \rangle$$

$$(2.82)$$

(vi) Lorentz symmetry:

$$U(\Lambda)|0\rangle = |0\rangle \qquad (2.83)$$

$\underset{\uparrow}{}$

(unitary) Fock space representation of Lorentz trafo Λ

one-particle states

$$|\vec{p}\rangle = \sqrt{2\omega_{\vec{p}}}\, a^{\dagger}(\vec{p})|0\rangle \qquad (2.85)$$

with $\langle\vec{q}|\vec{p}\rangle = \underbrace{2\omega_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q})}_{\text{Lorentz invariant}}$

and $U(\Lambda)|\vec{p}\rangle = |\Lambda\vec{p}\rangle$

- Lorentz invariance of $2\omega_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q})$:

Use Lorentz-invariant measure

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}} = \int\frac{d^4p}{(2\pi)^4}2\pi\,\delta(p^2-m^2)\,\Theta(p^0) \qquad (2.86)$$

invariant under proper, orthochronous Lorentz-trafos
$\det\Lambda = 1,\ (\Lambda^0{}_0 > 0)$ $SO(1,3)$

$$\Rightarrow \underbrace{\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}}_{\text{invariant}}\underbrace{2\omega_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q})}_{} = \underset{\uparrow}{1} \qquad (2.87)$$

$\underbrace{}\text{invariant} \longrightarrow \text{invariant} \Longleftarrow \text{invariant}$

Points (i) – (vi) complete Fock space construction

Remarks:

(i) $\phi(x)$ generates a superposition of one particle states from the vacuum,

$$\langle 0 | \phi(x) | \vec{p} \rangle = \langle 0 | \int \frac{d^3 p'}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{p}'}}} e^{-i p' x} a(\vec{p}') a^\dagger(\vec{p}) | 0 \rangle$$

$$= e^{-i p x} \qquad\qquad (2.88)$$

reminiscent of non-relativistic QM

$$\langle x | p \rangle = e^{-i p x}$$

(ii) Causality encoded in propagator

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

and its variants: Retarded, advanced

time-ordered props $\rightarrow$ chapter 3

perturb. theory

(3) Complex scalar field

$$S[\phi] = \int d^4x \left\{ \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi \phi^* \right\}$$

(2.89)

with $\qquad \phi = \frac{1}{\sqrt{2}} \left(\phi_1 + i \phi_2 \right)$

Field operator:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left\{ a(\vec{p}) e^{-ipx} + b^\dagger(\vec{p}) e^{ipx} \right\}$$

(2.90)

Canonical momentum: $\quad \Pi = \partial^0 \phi^*$

$$\Pi(x) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{p}}}{2}} \left\{ b(\vec{p}) e^{-ipx} - a^\dagger(\vec{p}) e^{ipx} \right\}$$

(2.91)

with $\qquad [\phi(\vec{x}), \Pi(\vec{y})] = \delta(\vec{x} - \vec{y})$

and
$$[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3 \delta(\vec{p} - \vec{q})$$

$$[b(\vec{p}), b^\dagger(\vec{q})] = (2\pi)^3 \delta(\vec{p} - \vec{q})$$

(2.92)

others vanish!

see Exercise

Hamiltonian:

$$H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left\{ \tilde{\pi}(\vec{p})\, \tilde{\pi}^*(\vec{p}) + \omega_{\vec{p}}^2\, \tilde{\phi}(\vec{p})\, \tilde{\phi}^*(\vec{p}) \right\}$$

$$\rightsquigarrow \quad H_{op} = \int \frac{d^3p}{(2\pi)^3}\, \omega_{\vec{p}} \left\{ a^+(\vec{p})\, a(\vec{p}) + b^+(\vec{p})\, b(\vec{p}) \right\}$$

sum of energy of part. and antipart. (2.93)

Charge: see p. 16 eqs (2.45), (2.47)

$$Q = i \int d^3x \left\{ \phi^* \partial_t \phi - \partial_t \phi^* \phi \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left\{ a^+(\vec{p})\, a(\vec{p}) - b^+(\vec{p})\, b(\vec{p}) \right\}$$

charge of antipart. (2.94)
≃ minus the charge of part.

and

$$\langle \alpha | Q | \alpha \rangle = \int \frac{d^3p}{(2\pi)^3} \left\{ \alpha^* \alpha(\vec{p}) - \beta^* \beta(\vec{p}) \right\}$$

(2.95)