3 Perturbation theory

3.1 Interaction picture

Fock space construction in the previous chapter 2.3 (c) in Heisenberg picture

$$i\partial_t |f\rangle = 0$$
$$i\partial_t \mathcal{O}(t) = [\mathcal{O}(t), H] \qquad (3.1)$$

with $\mathcal{O}(t) = e^{iHt}\,\mathcal{O}\,e^{-iHt}$

The field operator $\phi(x)$ indeed follows from $\phi(\vec{x})$ by $\phi(x) = e^{iHt}\phi(\vec{x})\,e^{-iHt}$:

real scalar: $$\phi(x) = e^{iHt}\underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left\{a(\vec{p})\,e^{i\vec{p}\vec{x}} + a^\dagger(\vec{p})\,e^{-i\vec{p}\vec{x}}\right\}}\,e^{-iHt}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left\{a(\vec{p})\,e^{-i\omega_{\vec{p}}t+i\vec{p}\vec{x}} + a^\dagger(\vec{p})\,e^{i\omega_{\vec{p}}t-i\vec{p}\vec{x}}\right\} \qquad (3.2)$$

with $$\boxed{e^{iHt}a(\vec{p})\,e^{-iHt} = a(\vec{p})\,e^{-i\omega_{\vec{p}}t}} \qquad (3.3)$$

Eq. (3.3) follows with

$$H\, a(\vec{p}) = a(\vec{p})\left(H - \omega_{\vec{p}}\right) \qquad (3.4)$$

with $H = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \omega_{\vec{p}}\, a^\dagger(\vec{p})\, a(\vec{p})$ & eqs. (2.57), (2.58)

$\leftarrow$ see eq. (2.63), p. 25

$$\Rightarrow e^{iHt}\, a(\vec{p})\, e^{-iHt} = a(\vec{p})\, e^{i(H-\omega_{\vec{p}})t}\, e^{-iHt}$$

$$= a(\vec{p})\, e^{-i\omega_{\vec{p}}t} \qquad (3.5)$$

Similarily: $e^{iHt}\, a^\dagger(\vec{p})\, e^{-iHt} = a^\dagger(\vec{p})\, e^{i\omega_{\vec{p}}t}$

Schrödinger picture:

$$i\partial_t\, |f(t)\rangle = H\,|f\rangle$$

$$i\partial_t\, \mathcal{O} = 0$$

with $|f(t)\rangle = e^{-iHt}\,|f\rangle$

Time evolution op. $\mathcal{U}(t,t') := e^{-iH(t-t')}$

acting either on states (Schrödinger) or

on operators (Heisenberg)

Remark: Causality is encoded in the op. $\phi(x)$:

$$[\phi(x), \phi(y)] = 0 \quad \text{for } (x-y)^2 < 0$$

(space-like)

(3.6)

This follows with, $\phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a(\vec{p})\, e^{-ipx} + a^\dagger(\vec{p}) e^{ipx} \right]$

see eq. (2.52), p. 19

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \frac{1}{\sqrt{2\omega_{\vec{q}}}} \Big\{ [a(\vec{p}), a^\dagger(\vec{q})] e^{-ipx+iqx}$$

$$+ [a^\dagger(\vec{p}), a(\vec{q})] e^{ipx-iqx} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-ip(x-y)} - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{ip(x-y)}$$

e.q. (2.87), p. 32 $\longrightarrow$

$$= \int \frac{d^4p}{(2\pi)^3} \delta(p^2-m^2)\, \Theta(p^0)\, e^{-ip(x-y)} \tag{3.7}$$

$$- \int \frac{d^4p}{(2\pi)^3} \delta(p^2-m^2)\, \Theta(p^0)\, e^{ip(x-y)}$$

$$\Rightarrow \quad \boxed{[\phi(x), \phi(y)] = 0 \quad \text{for } (x-y)^2 < 0} \tag{3.8}$$

Proof of (3.8):

$$\int \frac{d^4p}{(2\pi)^3}\, \delta(p^2-m^2)\,\Theta(p^0)\, e^{ip(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^3}\, \delta(p^2-m^2)\,\Theta(p^0)\, e^{ip\Lambda(x-y)} \tag{3.9}$$

with $\Lambda \in SO(1,3)$: $p^2>0$ & $p^0>0$: $(\Lambda p)^0>0$

For space-like $x-y$ there exist $\Lambda \in SO(1,3)$ with

$$\Lambda(x-y) = -(x-y)$$

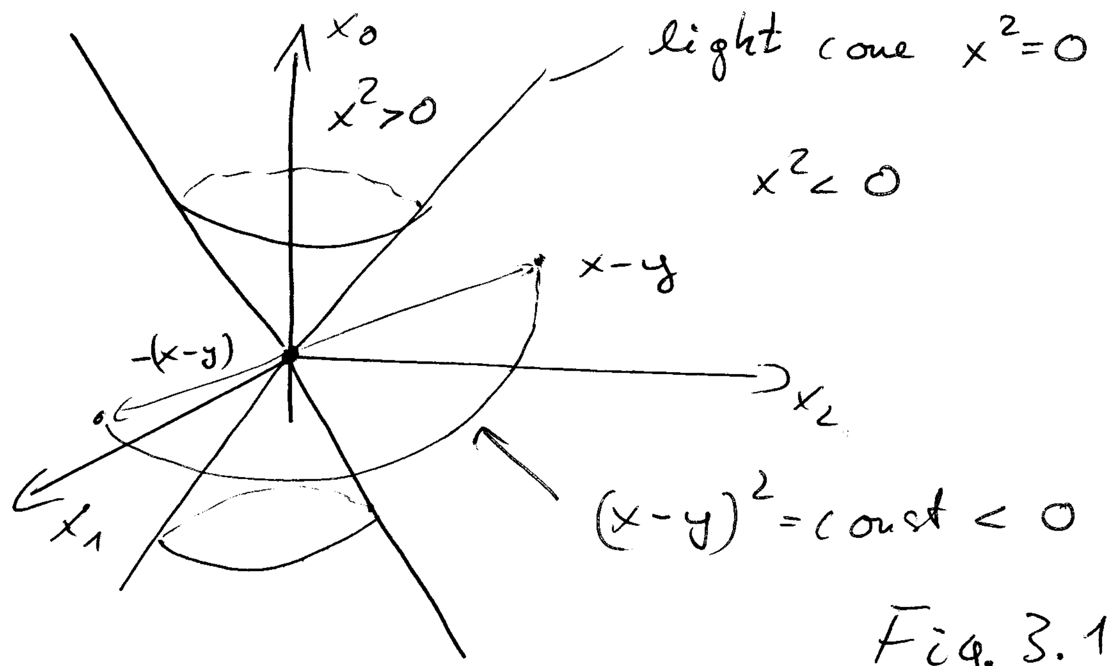
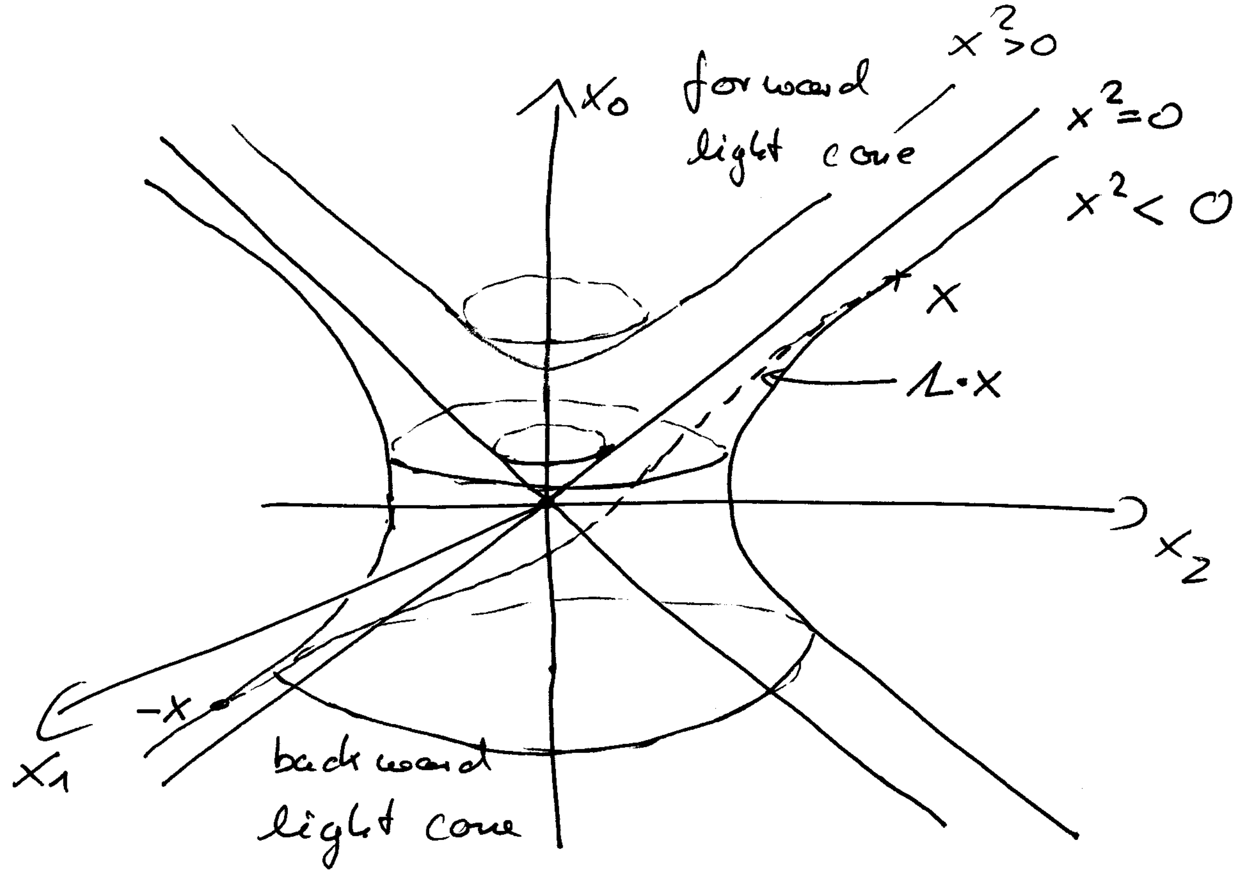


Fig. 3.1

Note that for $(x-y)^2>0$ there exist

<u>no</u> $\Lambda \in SO(1,3)$ with $\Lambda(x-y) = -(x-y)$!

Proof: See Fig. 3.1



- Lorentz trafos connect gen. two points on the $x^2 = \text{const.}$ surfaces

- For $x^2 > 0$ there are disconnected forward & backward branches.

- For $x^2 < 0$ there is a Lorentz trafo Λ

$x \to -x$ (or $(x-y) \to -(x-y)$)

[with cont. $\Lambda(s)$: $\Lambda(0) = \mathbb{1}$, $\Lambda(1) = \Lambda$]
(orthochron)

Lagrangian density:

$$\mathcal{L}(\phi) = \underbrace{\frac{1}{2}\phi(x)\left(-\partial^2 - m^2\right)\phi(x)}_{\mathcal{L}_0(\phi)} + \mathcal{L}_{int}(\phi) \qquad (3.10)$$

$\uparrow$ interaction

here $\mathcal{L}_{int}(\phi) = -V(\phi)$ polynomial in ϕ

$\Rightarrow$ Hamiltonian density:

$$\mathcal{H}(\Pi,\phi) = \underbrace{\frac{1}{2}\Pi(x)^2 + \frac{1}{2}\phi(x)\left[-\Delta + m^2\right]\phi(x)}_{\mathcal{H}_0(\Pi,\phi)} + \mathcal{H}_{int}(\phi) \qquad (3.11)$$

with $\mathcal{H}_{int}(\phi) = V(\phi)$

Hamiltonian bounded from below:

$$\boxed{V(\phi) = \frac{\lambda}{4!}\phi(x)^4} \qquad (3.12)$$

- higher terms excluded by renormalisability (in 4d)
- ϕ^3-term spoils sym. $\phi \to -\phi$ & boundedness
- ϕ^4-theory is 'working horse' of QFT

Standard method in QFT: perturbation theory

- consider $\lambda \ll 1$ and expand obervables, e.g. scattering amplitudes, in orders of λ: interaction is perturbation of free case!

$\Rightarrow$ Interaction picture:

- operators $\mathcal{O}(t)$ evolve in time with free Hamiltonian $H_0 = \int d^3x\, \mathcal{H}_0$

$$i\partial_t \mathcal{O} = [\mathcal{O}, H_0] \quad \Rightarrow \mathcal{O}(t) = e^{iH_0 t}\,\mathcal{O}\,e^{-iH_0 t} \tag{3.13}$$

- states $|f\rangle$ evolve with

$$i\partial_t |f\rangle = H_{int} |f\rangle \tag{3.14}$$

Note that $[H_0, H_{int}] \neq 0$!

$$\Rightarrow \partial_t H_{int} \neq 0 \quad : H_{int} = H_{int}(t)$$

Hence we have, $t > t_0$

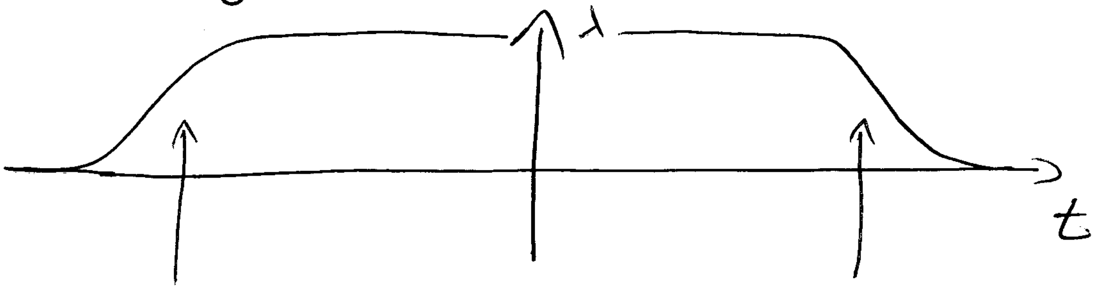
$$|f(t)\rangle = \mathcal{U}(t, t_0)\,|f(t_0)\rangle \tag{3.15}$$

with $\boxed{i\partial_t\,\mathcal{U}(t, t_0) = H_{int}(t)\,\mathcal{U}(t, t_0)}$

<u>Unitary</u> time-evolution op. defines the S-matrix

$$S = \lim_{\substack{t_0 \to -\infty \\ t \to +\infty}} \mathcal{U}(t, t_0) \tag{3.16}$$

Strictly speaking



λ is adiabatically switched on/off

$\Rightarrow$ $|\text{state } t \to -\infty\rangle = |i\rangle$ ← free

$|\text{state } t \to +\infty\rangle = |f\rangle$

Proper treatment: LSZ-formalism

Construction of $\mathcal{U}(t,t_0)$:

Infinitesimal form of eq. (3.14), p. 41

$$|f(t+\Delta t)\rangle = |f(t)\rangle - i\Delta t H_{int}(t)|f(t)\rangle$$
$$= (1 - i\Delta t H_{int}(t))|f(t)\rangle$$
$$= (1-i\Delta t H_{int}(t))(1-i\Delta t H_{int}(t-\Delta t))|f(t-\Delta t)\rangle$$

Iteration: (3.17)

$$\Rightarrow |f(t+\Delta t)\rangle = \underbrace{\prod_{n=0}^{N}(1-i\Delta t H_{int}(t-n\Delta t))}_{\mathcal{U}(t+\Delta t,\, t-N\Delta t)}|f(t-N\Delta t)\rangle \qquad (3.18)$$

Expansion in powers of Δt:

$$\mathcal{U}(t+\Delta t, t-N\Delta t) = 1 + (-i)\Delta t \sum_{n=0}^{N} H_{int}(t-n\Delta t)$$
$$+ (-i)^2(\Delta t)^2 \sum_{n<m} H_{int}(t-n\Delta t)H_{int}(t-m\Delta t)$$
$$+ \ldots$$

$\Delta t \to 0$ with $N\Delta t = t - t_0$

$$\Rightarrow 1 + (-i)\int_{t_0}^{t} dt' H_{int}(t')$$
$$+ (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_{int}(t')H_{int}(t'')$$
$$+ \ldots \qquad (3.19)$$

Finally

$t > t_0$: $$\boxed{\mathcal{U}(t, t_0) = T \exp\left\{-i \int_{t_0}^{t} dt' H_{int}(t')\right\}}$$

(3.20)

with $T\, A(t)\, B(t') = A(t)\, B(t')\, \Theta(t-t')$

$+ B(t')\, A(t)\, \Theta(t'-t)$

For example:

$$\frac{1}{2} T \int_{t_0}^{t} dt'\, H_{int}(t') \int_{t_0}^{t} dt''\, H_{int}(t'')$$

$$= \frac{1}{2}\left\{\int_{t_0}^{t} dt'\, H_{int}(t') \int_{t_0}^{t'} dt''\, H_{int}(t'')\right.$$

$$\left. + \int_{t_0}^{t} dt''\, H_{int}(t'') \int_{t_0}^{t''} dt'\, H_{int}(t')\right\}$$

$$= \int_{t_0}^{t} dt'\, H_{int}(t') \int_{t_0}^{t'} dt''\, H_{int}(t'') \qquad (3.21)$$

Remark: $H_{int} = \int d^3x\, \Phi^4(x) \sim a^2 a^{+2}$

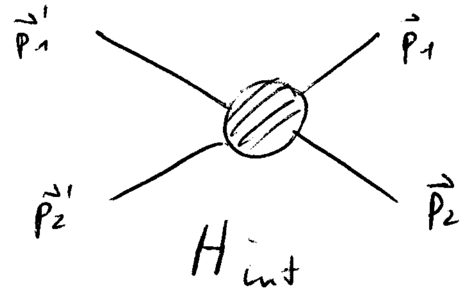
creates 2 particles and annihilates

them : $\langle 0 | H_{int} | 0 \rangle$ infinite

vacuum processes

Example: 2 to 2 scattering

$\langle \vec{p}_1{}' \, \vec{p}_2{}' | \sim$ at $t \to +\infty$



$\sim | \vec{p}_1 \, \vec{p}_2 \rangle$ at $t \to -\infty$

no scattering

S-matrix : $S = \mathbb{1} + iT$

$O(\lambda)$

In our case

$$iT_{fi} \underset{\uparrow}{\simeq} -i \langle 0 | a(\vec{p}_1{}') \, a(\vec{p}_2{}') \, \lambda/4! \int d^4x \, \phi^4(x) \, a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2) | 0 \rangle$$

infinities

with $\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left\{ a(\vec{p}) e^{-ipx} + a^\dagger(\vec{p}) e^{ipx} \right\}$ (3.22)

eq.(2.57) p.21

We use $[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3 \delta(\vec{p} - \vec{q})$ to pull the a's in H_{int} to the right: $\int d^4x \, \phi^4(x) \sim a^{\dagger 2} a^2$

$$\Rightarrow iT_{fi} \simeq -i\lambda \left[(2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \right]$$ (3.23)

Matrix element $\mathcal{M}$ energy-momentum conservation

$$iT_{fi} =: i\mathcal{M} \, (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')$$

with $i\mathcal{M} = -i\lambda$

Remarks:

(i) Normal ordering:

$$: a(\vec{p}_1)\, a^\dagger(\vec{p}_2) : \; = \; a^\dagger(\vec{p}_2)\, a(\vec{p}_1)$$
(3.24)

e.g. Hamiltonian in free scalar theory

eq. 2.62, p. 23:

$$:H_0: \; = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} : \frac{1}{2} a^\dagger(\vec{p}) a(\vec{p}) + \frac{1}{2} a(\vec{p}) a^\dagger(\vec{p}) :$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \; a^\dagger(\vec{p})\, a(\vec{p}) \qquad (3.26)$$

no infinite vacuum terms

(ii) Normal ordered Hint already gives eq. (3.23):

$$\lambda/4! \int d^4x : \phi^4(x) : \; \sim \lambda/4! \; : a^{\dagger 2} a + a^\dagger a a^\dagger a + a a^\dagger a a^\dagger$$
$$+ a^\dagger a^2 a^\dagger + a a^{\dagger 2} a + a^2 a^{\dagger 2} :$$

$$\sim \lambda/4 \; a^{\dagger 2} a^2 \qquad (3.26)$$

(iii) Difference gives 'vacuum contributions'

$$H_{int} = :H_{int}: + \lambda/8 \int d^4x \left(\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \right)^2 \qquad (3.27)$$
$$+ (a^\dagger a, a a^\dagger)\text{-terms} \qquad [a, a^\dagger]^2 \searrow O^2$$

Computation of (3.23) : eq.(3.26)

$$\frac{\lambda}{4}\left[\prod_i \int \frac{d^3q_i}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec q_i}}}\right] e^{-ix(q_3+q_4-q_1-q_2)} \langle 0|\, a(\vec p_1{}')\,a(\vec p_2{}')\left[a^\dagger(\vec q_1)\,a^\dagger(\vec q_2)\right.$$

$$\left.\cdot\, a(\vec q_3)\,a(\vec q_4)\right] a^\dagger(\vec p_1)\,a^\dagger(\vec p_2)|0\rangle$$

$$\cdot\sqrt{2\omega_{\vec p_1{}'}\,2\omega_{\vec p_2{}'}\,2\omega_{\vec p_1}\,2\omega_{\vec p_2}}$$

$$= 4\cdot\frac{\lambda}{4}\left[\prod_i \int \frac{d^3q_i}{(2\pi)^3}\sqrt{\frac{2\omega_{\vec q_i}}{2\omega_{\vec q_i}}}\;\delta(\vec p_1{}'-\vec q_1)\,\delta(\vec p_2{}'-\vec q_2)\,\delta(\vec p_1-\vec q_3)\,\delta(\vec p_2-\vec q_4)\right.$$

$$\cdot\, e^{-ix(q_3+q_4-q_1-q_2)}$$

$$= \lambda\, e^{-ix(p_1+p_2-p_1'-p_2')}$$

with e.g.

$$a(\vec q_4)\,a^\dagger(p_1)\,a^\dagger(p_2)|0\rangle = \left[a(\vec q_4),\, a^\dagger(\vec p_1)\right] a^\dagger(\vec p_2)|0\rangle$$

$$+\,a^\dagger(\vec p_1)\,a(\vec q_4)\,a^\dagger(\vec p_2)\,|0\rangle$$

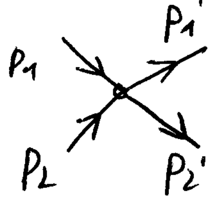
$$= \Big\{(2\pi)^3\delta(\vec q_4-\vec p_1)\,a^\dagger(\vec p_2) + a^\dagger(\vec p_1)\left[a(\vec q_4),\, a^\dagger(\vec p_2)\right]$$

$$+\, a^\dagger(\vec p_1)\,a^\dagger(\vec p_2)\underbrace{a(\vec q_4)\Big\}|0\rangle}_{0}$$

$$= (2\pi)^3\left[\delta(\vec q_4-\vec p_1)\,a^\dagger(\vec p_2) + \delta(\vec q_4-\vec p_2)\,a^\dagger(\vec p_1)\right]$$

$\vec q_3, \vec q_4$ integrat. variable : under integral: $2\cdot 2\ \overset{2}{(2\pi)^3}\ \delta(\vec q_4-\vec p_2)\,\delta(\vec q_3-\vec p_1)$
with $a(\vec q_3)$ ----

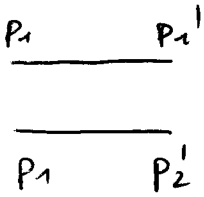
Interpretation:

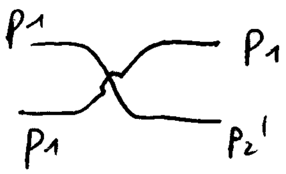


: $-i\lambda \quad \cdot \quad (2\pi)^4 \delta(p_1+p_2-p_1'-p_2')$

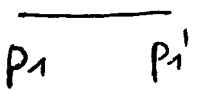
interaction strength   4-momentum conservation

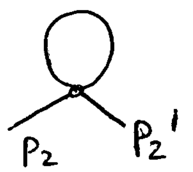
vacuum parts: $\langle 0| a(\vec{p}_1{}')\, a(\vec{p}_2{}')\, a^\dagger(\vec{p}_1)\, a^\dagger(\vec{p}_2)|0\rangle$

$-i\lambda \Bigg($ 

$+$ 

$\Bigg) \cdot \int d^4x\, O^2 \leftarrow$ ,

$-i\lambda \Big[$ 

$\cdot$ 

$+ (p_1 \leftrightarrow p_2) + (p_1' \leftrightarrow p_2') + (p_1 \leftrightarrow p_2,\, p_1' \leftrightarrow p_2') \Big]$

loop contribution

First term:

$$\langle 0|\, a(\vec{p}_1{}')\, a(\vec{p}_2{}')\, a^\dagger(\vec{p}_1)\, a^\dagger(\vec{p}_2)|0\rangle \underbrace{\left(1 - i\lambda \int d^4x\, O^2\right)}_{\exp\{-i\lambda\int d^4x\, O^2\} + O(\lambda^2)}$$

Phase/Loops are infinite: call for appropriate treatment

Core example:

two-point fct.: propagator

(i) Start in Heisenberg picture:

vacuum of full theory: $|\Omega\rangle$

with $i\partial_t |\Omega\rangle = 0$ (3.28)

Operators evolve with full Hamiltonian, i.e.

$$i\partial_t \phi_H = [\phi_H, H] \quad (3.29)$$
$$\text{with } \phi_H = e^{-iHt}\phi(0,\vec{x})e^{iHt}$$

Link to interaction picture:

interaction picture states $|f\rangle_I$ evolve with H_{int}:

$$|f(t)\rangle_I = \mathcal{U}(t,0)\,|f(0)\rangle_I \quad (3.30)$$

operators

$$i\partial_t \phi_I = [\phi_I, H_0] \quad (3.31)$$

Hence, ϕ_I have the representation eq. (2.52), p. 19

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left[a(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{ipx}\right]_{p_0=\omega_{\vec{p}}}$$

It follows: (4th tutorial: $\mathcal{U}(t,0) = e^{iH_0 t} e^{-iHt}$)

$$\phi_H(x) = \mathcal{U}(0,x^0)\,\phi_I(x)\,\mathcal{U}(x^0,0) \qquad (3.32)$$

with $\quad \phi_H(x)|f\rangle_H = \mathcal{U}(0,x^0)\,\phi_I(x)\,\mathcal{U}(x^0,0)|f\rangle_H$

and $\quad i\partial_t \mathcal{U}(x^0,0)|f\rangle_H = H_{int}\,\mathcal{U}(x^0,0)|f\rangle_H$

It is tempting to identify $\mathcal{U}(x^0,0)|f\rangle_H$ with the interaction picture states $|f(t)\rangle_I$.

At $t \to \pm\infty$, λ is switched off adiabtically, and $|f\rangle_I$ tend to free in/out states. We have

$$\langle\Omega|\mathcal{U}(0,x^0) = \langle\Omega|\underbrace{\mathcal{U}(0,\infty)}_{\mathcal{U}(\infty,0)^{-1}}\mathcal{U}(\infty,x^0)$$

adiabaticity: $|n\text{-part.}\rangle_{free}(t=\infty) \xrightarrow{\mathcal{U}} |n\text{-particle}\rangle_{full}$

$$= \sum_n \langle\Omega|\mathcal{U}(0,\infty)|n\rangle_I\, {}_I\langle n|\mathcal{U}(\infty,x^0)$$

$$= \underbrace{\langle\Omega|\mathcal{U}(0,\infty)|0\rangle}_{\text{Please, see p. 51a}}\langle 0|\mathcal{U}(\infty,x^0)$$

Also: $\quad \mathcal{U}(x^0,0)|\Omega\rangle = \mathcal{U}(x^0,-\infty)|0\rangle\langle 0|\mathcal{U}(-\infty,0)|\Omega\rangle$

$$\mathcal{U}(t,0) = e^{iH_0 t}\, e^{-iHt} \quad , \; H = H_0 + H_{int}$$

with

$$i\partial_t \mathcal{U}(t,0) = H_I(t)\, \mathcal{U}(t,0)$$

where

$$\underset{H_{int}(t)}{\overset{\shortparallel}{H_I(t)}} = e^{iH_0 t} H_{int}\, e^{-iH_0 t} = \frac{\lambda}{4!}\int d^3x\, \phi_I^4(x)$$

with

$$i\partial_t H_I = [H_I, H_0]$$

It follows that

$$\begin{aligned} i\partial_t \phi_H(x) = \; & \mathcal{U}(0,t)\, H_{int}(x^0)\, \phi_I(t)\, \mathcal{U}(t,0) \\ & - \mathcal{U}(0,t)\, \phi_I\, H_{int}(\;)\, \mathcal{U}(t,0) \\ & + \mathcal{U}(0,t)\, [\phi_I(x), H_{int}(x)]\, \mathcal{U}(t,0) \\ = \; & 0 \end{aligned}$$

Propagator:

$$\langle \Omega | T \phi_H(x)\, \phi_H(y) | \Omega \rangle$$

$$= \langle \Omega | \phi_H(x)\, \phi_H(y) | \Omega \rangle\, \Theta(x^0 - y^0)$$
$$+ \langle \Omega | \phi_H(y)\, \phi_H(x) | \Omega \rangle\, \Theta(y^0 - x^0) \qquad (3.35)$$

For $x^0 > 0 > y^0$:

$$\langle \Omega | \phi_H(x)\, \phi_H(y) | \Omega \rangle$$

(3.32) $\longrightarrow$
$$= \langle \Omega | \mathcal{U}(0,x^0)\, \phi_I(x)\, \overset{\mathcal{U}(x^0,0)\,\mathcal{U}(0,y^0)}{\overset{\|}{\mathcal{U}(x^0,y^0)}}\, \phi_I(y)\, \mathcal{U}(y^0,0) | \Omega \rangle$$
$$= \langle 0 | \mathcal{U}(\infty,x^0)\, \phi_I(x)\, \mathcal{U}(x^0,y^0)\, \phi_I(y)\, \mathcal{U}(y_0,-\infty) | 0 \rangle$$
$$1/\left(\left(\langle \Omega | \mathcal{U}(0,\infty) | 0 \rangle\, \langle 0 | \mathcal{U}(-\infty,0) | \Omega \rangle\right)^{-1} \qquad (3.36)$$

where we have used that in general

$$\mathcal{U}(x^0, y^0)\, \mathcal{U}(y^0, z^0) = \mathcal{U}(x^0, z^0) \qquad (3.37)$$

for $x^0 > y^0 > z^0$. This follows straightforwardly from eq. (3.20), p. 44.

Note that the denominator in (3.36) is (a product of two) phases, see p.51a. We again use the adiabaticity, as in (3.34), and get

$$\langle \Omega | \mathcal{U}(0,\infty) | 0 \rangle^{-1} \langle 0 | \mathcal{U}(-\infty,0) | \Omega \rangle^{-1}$$

p.51a $\to$ $$= \langle 0 | \mathcal{U}(\infty,0) | \Omega \rangle \langle \Omega | \mathcal{U}(0,-\infty) | 0 \rangle$$

(3.37) $\to$ $$= \langle 0 | \mathcal{U}(\infty,-\infty) | 0 \rangle$$

$$= \langle 0 | T \exp\left\{ -i \int dt\, H_{int}(t) \right\} | 0 \rangle \qquad (3.38)$$

We also have for the numerator of (3.36)

$$\langle 0 | \mathcal{U}(\infty, x^0)\, \phi_I(x)\, \mathcal{U}(x^0, y^0)\, \phi_I(y)\, \mathcal{U}(y^0,-\infty) | 0 \rangle$$

$$= \langle 0 | T\, \phi_I(x)\, \phi_I(y) \exp\left\{ -i \int dt\, H_{int}(t) \right\} | 0 \rangle \qquad (3.37)$$

Finally, with $\phi_I = \phi$, and the analogous result for $y_0 > x_0$,

$$\boxed{\langle \Omega | T\, \phi_H(x)\, \phi_H(y) | \Omega \rangle = \frac{\langle 0 | T\, \phi(x)\, \phi(y) \exp\left\{ -i \int dt\, H_{int}(t) \right\} | 0 \rangle}{\langle 0 | T \exp\left\{ -i \int dt\, H_{int}(t) \right\} | 0 \rangle}} \qquad (3.38)$$

$|\langle\Omega|U(0,\infty)|0\rangle| = 1$:

(i) $|\langle\Omega|U(0,x^0)| = 1$ from U being unitary:

$$|\langle\Omega|U(0,x^0)|^2 = \langle\Omega|U(0,x^0)U^\dagger(0,x^0)|\Omega\rangle$$
$$= \langle\Omega|\Omega\rangle = 1$$

(ii) $|\langle 0|U(-\infty,x^0)|^2 = 1$ from U being unitary and $\langle 0|0\rangle = 1$

From (i) and (ii) it follows that

$|\langle\Omega|U(0,\infty)|0\rangle| = 1$. Hence

$$\langle\Omega|U(0,\infty)|0\rangle^{-1} = \langle\Omega|U(0,\infty)|0\rangle^*$$
$$= \langle 0|U^\dagger(0,\infty)|\Omega\rangle$$

Eq. (3.38) is straightforwardly extended to

$$\langle\Omega|T\phi_H(x_1)\cdots\phi_H(x_n)|\Omega\rangle$$

$$=\frac{\langle 0|T\,\phi(x_1)\cdots\phi(x_n)\,e^{-i\int dt\,H_{int}}|0\rangle}{\langle 0|T\,e^{-i\int dt\,H_{int}}|0\rangle} \qquad (3.39)$$

Remarks:

(i) The denominator in (3.38), (3.39) is a phase. For example, the linear term in λ is

$$-i\langle 0|\int dt\,H_{int}|0\rangle=-i\lambda\langle 0|\int d^4x\,\phi^4|0\rangle$$

$$=-i/8\int d^4x\left(\int\frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_{\vec q}}\right)^2 \qquad (3.40)$$

$\Rightarrow$ cancels vacuum term in (3.27)

(ii) The phase factor is infinite, as are the vacuum contributions in the numerator: the infinities cancel!