3.2 Wick's theorem

We have seen, that the computation of scattering amplitudes relates to that of time-ordered n-point fcts.:

$$\langle 0 | T \phi(x_1) \cdots \phi(x_n) \, e^{i\int d^4y \, \mathcal{L}_{int}(y)} | 0 \rangle$$

where $-\int dt\, H_{int} = \int dt\, L_{int} = \int d^4y\, \mathcal{L}_{int}(y)$, and the coupling $\lambda \ll 1$. Since

$$\langle 0 | T \phi(x_1) \cdots \phi(x_n) \prod_{i=1}^{m} \mathcal{L}_{int}(y_i) | 0 \rangle$$

$$= \frac{1}{(4!)^m} \langle 0 | \; T\phi_1(x_1) \cdots \phi(x_n)\, \phi(x_{n+1}) \cdots \phi(x_{n+4m}) | 0 \rangle$$

with $x_{n+1}, \dots, x_{n+4} = y_1 ; \dots ; x_{n+4(m-1)}, \dots, x_{n+4m} = y_m$

the only building block is

$$\boxed{\langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle} \quad (3.41)$$

For $x_1^0 > x_2^0 > \cdots > x_n^0$, eq. (3.41) boils down to

$$\langle 0 | \phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

and we simply have to use the canonical commutation relations for a, a^+ (Note that $\phi = \phi_I$ is free!).

Strategy: write $T\phi(x_1)\cdots\phi(x_n)$ contains $:\phi(x_1)\cdots\phi(x_{n-2}):$ $\cdots$

as $:\phi(x_1)\cdots\phi(x_n): +$ ~~rest~~. Taking the vaccum expectation value, the normal ordered part vanishes!

Example: two-point fct.

$$\phi(x) = \phi_+(x) + \phi_-(x)$$

with $\phi_+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} a^+(\vec{p})\, e^{ipx}$

creation

annihilation $\longrightarrow$ $\phi_-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} a(\vec{p})\, e^{-ipx}$

$$(3.42)$$

For $x^0 > y^0$:

$$T\,\phi(x)\phi(y) = \phi_+(x)\phi_+(y) + \phi_+(x)\phi_-(y) + \phi_-(x)\phi_+(y) + \phi_-(x)\phi_-(y)$$

$$= \phi_+(x)\phi_+(y) + \phi_+(x)\phi_-(y) + \phi_+(y)\phi_-(x) + \phi_-(x)\phi_-(y)$$
$$+ [\phi_-(x), \phi_+(y)]$$

$$(3.43)$$

$$\Rightarrow \; T\,\phi(x)\,\phi(y)\Big|_{x_0>y_0} = \; :\phi(x)\,\phi(y): + [\phi_-(x), \phi_+(y)] \tag{3.44}$$

where $:\phi_-(x)\,\phi_+(y): \; = \phi_+(y)\,\phi_-(x) \quad \forall x$

from $:a(\vec{p})\,a^+(\vec{q}): \; = a^+(\vec{q})\,a(\vec{p})$ (3.45)

- Taking vacuum expectation values, the normal ordered part vanishes:

$$\boxed{\begin{aligned}\langle 0|\,T\,\phi(x)\,\phi(y)\,|0\rangle &= [\phi_-(x), \phi_+(y)]\,\Theta(x^0-y^0)\\ &\quad + [\phi_-(y), \phi_+(x)]\,\Theta(y^0-x^0)\end{aligned}} \tag{3.46}$$

The time-ordered propagator is called Feynman propagator:

$$D_F(x-y) = \langle 0|\,T\,\phi(x)\,\phi(y)\,|0\rangle \tag{3.47}$$

It is the key-ingredient in (time-ordered) perturbation theory.

Computation of D_F :

$$[\phi_-(x), \phi_+(y)]\,\Theta(x^0-y^0)$$

$$= \int \frac{d^3q}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec p}}\frac{1}{2\omega_{\vec q}}\left[a(\vec p), a^\dagger(\vec q)\right] e^{-ipx+iqy}\,\Theta(x^0-y^0)$$

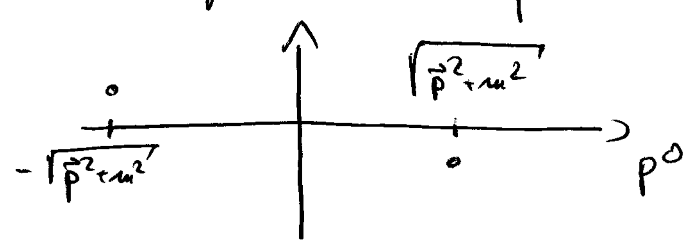
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec p}}\, e^{-ip(x-y)}\,\Theta(x^0-y^0)$$

$$\Rightarrow \quad D_F(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec p}}\left\{e^{-ip(x-y)}\Theta(x^0-y^0) + e^{ip(x-y)}\Theta(y^0-x^0)\right\} \tag{3.48}$$

D_F can be rewritten as $(\varepsilon \to 0_+)$

$$\boxed{D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}\, e^{-ip(x-y)} \qquad (3.49)}$$

Proof : The integrand has poles at $p^0 = \pm\sqrt{\vec p^{\,2}+m^2-i\varepsilon}$



$x^0-y^0>0$: close contour in lower half plane

$x^0-y^0<0$: " " " upper " "

$x^0 - y^0 > 0$: pole at $p^0_- = \sqrt{\vec{p}^2 + m^2 - i\varepsilon} \to \omega_{\vec{p}}$

$$D_F(x-y) = -\int \frac{d^3p}{(2\pi)^3} \frac{2\pi i}{2\pi} \operatorname*{res}_{p^0_-} \left[\frac{e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \, i \, \frac{e^{-ip(x-y)}}{2i\omega_{\vec{p}}} \Bigg|_{p^0 = \omega_{\vec{p}}} \qquad (3.48)$$

Similarily for $x^0 - y^0 < 0$. $\square$

Remark:

(i) We have parameterised the time-ordered propogator in terms of commutators. On operator level we have

$$T\,\phi(x)\phi(y) = :\phi(x)\phi(y): \qquad (3.49)$$
$$+ [\phi_-(x), \phi_+(y)]\,\Theta(x^0 - y^0)$$
$$+ [\phi_-(y), \phi_+(x)]\,\Theta(y^0 - x^0)$$

or $$T\,\phi(x)\,\phi(y) = :\phi(x)\,\phi(y): + \underbrace{\overline{\phi(x)\,\phi(y)}}_{\text{contraction}} \qquad (3.50)$$

where

$$\overline{\phi(x)\,\phi(y)} = [\phi_-(x), \phi_+(y)]\,\Theta(x^0-y^0)$$
$$+ [\phi_-(y), \phi_+(x)]\,\Theta(y^0-x^0) \qquad (3.51)$$
$$= D_F(x-y) \leftarrow \text{c-number}$$

Generalisation to product of n fields:

Wick's Theorem:

$$\boxed{T\,\phi(x_1)\cdots\phi(x_n) = :\phi(x_1)\cdots\phi(x_n) + \text{all contractions}:} \qquad (3.52)$$

where

$$\phi(x_1)\cdots\overline{\phi(x_i)\cdots\phi(x_j)}\cdots\phi(x_n)$$
$$= \phi(x_1)\cdots\phi(x_{i-1})\phi(x_{i+1})\cdots\phi(x_{j-1})\phi(x_{j+1})\cdots\phi(x_n)\;\overline{\phi(x_i)\phi(x_j)}$$

Example:

(1) $$T\,\phi(x_1)\,\phi(x_2) = :\phi(x_1)\,\phi(x_2) + \wick{\c\phi(x_1)\,\c\phi(x_2)}$$

(2) 4-point correlation function

$$T\,\phi(x_1)\cdots\phi(x_4) = T\,\phi_1\,\phi_2\,\phi_3\,\phi_4$$

$$\begin{aligned} = \,:\,&\phi_1\phi_2\phi_3\phi_4 + \wick{\c\phi_1\c\phi_2}\phi_3\phi_4 + \phi_1\phi_2\wick{\c\phi_3\c\phi_4} + \wick{\c\phi_1\phi_2\c\phi_3}\phi_4 \\ &+ \wick{\c\phi_1\phi_2\phi_3\c\phi_4} + \phi_1\wick{\c\phi_2\c\phi_3}\phi_4 + \phi_1\wick{\c\phi_2\phi_3\c\phi_4} \\ &+ \wick{\c\phi_1\c\phi_2}\wick{\c\phi_3\c\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}\,: \end{aligned} \qquad (3.53)$$

where e.g.

$$:\wick{\c\phi_1\c\phi_2}\phi_3\phi_4: = :\phi_3\phi_4:\,\wick{\c\phi_1\c\phi_2} = :\phi_3\phi_4:\,D_F(x_1-x_2) \qquad (3.54)$$

Note also that

$$\langle 0|:\mathcal{O}:|0\rangle = 0 \qquad (3.55)$$

It follows with eq. (3.53) and eq. (3.55) that

$$\langle 0|\,T\,\phi(x_1)\cdots\phi(x_4)|0\rangle \qquad (3.56)$$

$$= D_F(x_1-x_2)\,D_F(x_2-x_3) + D_F(x_1-x_3)\,D_F(x_2-x_4)$$

$$+ D_F(x_1-x_4)\,D_F(x_2-x_3)$$

Proof of Wick's theorem (by induction)

(i) $n = 1, 2$: $T\,\phi_1 = :\phi_1:$ ✓

$$T\,\phi_1\phi_2 = :\phi_1\phi_2: + \phi_1^{\sqcap}\phi_2^{\sqcap}$$ ✓

(ii) Assume, Wick's theorem applies to

$T\,\phi_2 \cdots \phi_{n+1}$. Without loss of gen.: $\boxed{x_1^0 \geq x_i \;\forall i}$

Then $T\,\phi_1 \cdots \phi_{n+1}$

$$= \phi_1 \left(:\phi_2 \cdots \phi_{n+1} + \text{all contractions}: \right)$$

$$= (\phi_{1+} + \phi_{1-}) \left(:\phi_2 \cdots \phi_{n+1} + \text{all contr.}: \right)$$

$$= :\phi_1 \cdots \phi_{n+1} + [\phi_{1-}, \phi_2]\, \phi_3 \cdots \phi_{n+1}$$

$$+ \phi_2 [\phi_{1-}, \phi_3]\, \phi_4 \cdots \phi_{n+1} + \cdots + \phi_2 \cdots [\phi_{1-}, \phi_{n+1}]:$$

$$+ (\phi_{1+} + \phi_{1-}) (:\text{all contractions}:) \qquad (3.57)$$

We use

$$[\phi_{1-}, \phi_i] = [\phi_{1-}, \phi_{i+}] = \phi_1^{\sqcap}\phi_i^{\sqcap} \qquad (3.58)$$

and similarly as in (3.57) for

$$(\phi_{1+} + \phi_{1-}) (:\text{all contr.}:)$$

Then

$$T\,\phi_1 \cdots \phi_{n+1} = :\phi_1 \cdots \phi_{n+1} + \text{all contr.}: \qquad \blacksquare$$