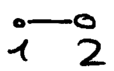
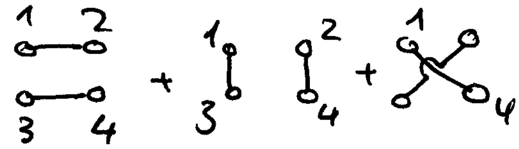
3.3 Feynman rules

With Wick's theorem we write every time ordered n-point fct. as product of Feynman propagators. We introduce the diagrammatical notation

$$D_F(x_1 - x_2) = \langle 0 | T\, \phi_1 \phi_2 | 0 \rangle =$$ 

It follows e.g.:

$$\langle 0 | T\, \phi_1 \cdots \phi_4 | 0 \rangle =$$ 

What about $\langle 0 | T\, \phi_1 \phi_1 | 0 \rangle$?

$$D_F(0) =$$ 

$D_F(x)$ was given on p. 56, eq. (3.49):

$$\curvearrowright \quad D_F(0) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} \qquad (3.59)$$

is singular. As already done before, this singularity will be removed by an appropriate adjustment of our computation (renormalisation).

Remark: $\left[\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} \right] = 2$

$$\Rightarrow \mathcal{D}_F(0) = \mathcal{M}^2 + \text{infinite}$$

Relevant example: 2-2 scattering:

(i) $O(\lambda^0)$: $\langle 0 | T \phi_1 \cdots \phi_4 | 0 \rangle$

(ii) $O(\lambda)$: $\phi = \phi(x)$

$$-\frac{i\lambda}{4!}\int d^4x \langle 0 | \phi_1 \cdots \phi_4 \phi\phi\phi\phi | 0\rangle$$

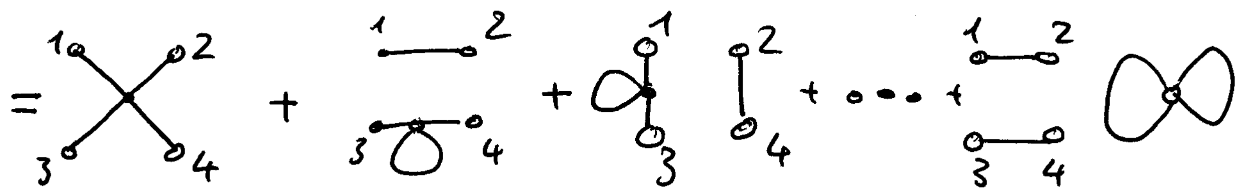
$$= -\frac{i\lambda}{4!}\int d^4x \Big[\overline{\phi_1 \phi}\ \overline{\phi_2 \phi}\ \overline{\phi_3 \phi}\ \overline{\phi_4 \phi} \cdot 4!$$

← # of pos. to distribute ϕ^4 to $\phi_1\phi_2\phi_3\phi_4$

$$+ \overline{\phi\phi}\left\{ \overline{\phi_1 \phi}\ \overline{\phi_2 \phi}\ \overline{\phi_3 \phi_4} + \text{permut.}\right\}\cdot 12$$

$$+ \overline{\phi\phi}\ \overline{\phi\phi}\left\{ \overline{\phi_1 \phi_2}\ \overline{\phi_3 \phi_4} + \text{permut}\right\}\cdot 3 \Big]$$

Diagrammatically (without sym. fact.)



+ ...

where  stands for $[-i\lambda \int d^4x]$

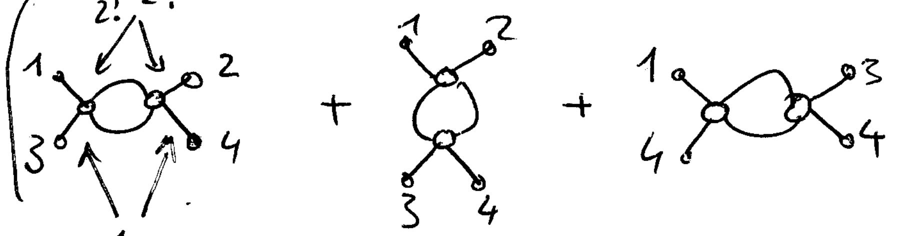
(iii) $O(\lambda^2)$:

$$\left(\frac{-i\lambda}{4!}\right)^2 \int d^4x \int d^4z \langle 0 | \phi_1 \cdots \phi_4\, \phi^4(x)\, \phi^4(z) | 0\rangle$$

Diagrammatically:

$1/S$ $\quad \frac{1}{2!}\,2!$

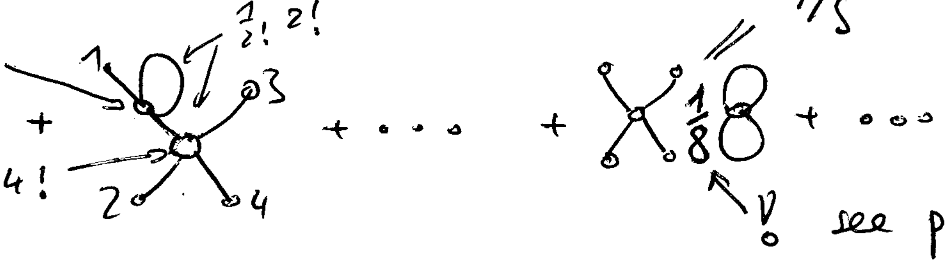
$$O(\lambda^2) = \frac{1}{2}\Bigg($$



$$\Bigg)$$

$\frac{1}{4!}\,4!$

$\frac{1}{S} = \frac{1}{4!}\cdot 4\cdot 3$



see p. 45 eq. 46

Combinatorics: Taylorexpansion $\frac{1}{n!}H_{int}^n$

(i) $\frac{1}{S}\;\frac{1}{4!}\;4!\;\frac{1}{n!}\;n! \leftarrow$ permut. of $\underbrace{H_{int}\cdots H_{int}}_{n\text{-times}}$

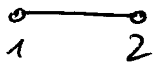
prefactor in H_{int} $\quad$ permutations of how to contract ϕ^4 in H_{int} with 4 <u>other</u> ϕ's

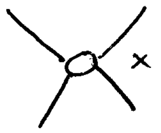
(ii) Symmetry factor S:

Simple comb. of (i) is changed if loops are present: ϕ's in ϕ^4 cannot be counted as "other ϕ's"

$\Rightarrow$ $1/S$ with $S = \#$ of interchanging components without changing diagram

Feynman rules: comp. of $\langle 0|T\phi_1\dots\phi_n e^{-i\int d^4x\,\phi^4}|0\rangle$

(i)  $= D_F(x_1-x_2)$

(ii)  $= (-i\lambda)\int d^4x$

(iii) multiplication with $1/S$

Use for final result $\quad (\langle\Omega|\mathcal{O}|\Omega\rangle =: \langle\mathcal{O}\rangle)$

$$\frac{\langle 0|T\phi_1\dots\phi_n\, e^{i\int d^4x\,\mathcal{L}_{int}}|0\rangle}{\langle 0|T e^{i\int d^4x\,\mathcal{L}_{int}}|0\rangle} =: \langle T\phi_1\dots\phi_n\rangle$$

For the comput. we note that the terms $\langle 0|T\phi_1\dots\phi_n\,\mathcal{L}_{int}{}^m/m!|0\rangle$ can be ordered in terms of contractions between the ϕ_i and the $\mathcal{L}_{int}$'s:

$$\langle 0|T\phi_1\dots\phi_n\,\mathcal{L}_{int}{}^m/m!|0\rangle = \langle 0|T\phi_1\dots\phi_n|0\rangle\langle 0|T\mathcal{L}_{int}{}^m|0\rangle\frac{1}{m!}$$
$$+\langle 0|T\phi_1\dots\phi_n\,\mathcal{L}_{int}|0\rangle\langle 0|T\mathcal{L}_{int}{}^{m-1}|0\rangle\frac{1}{(m-1)!}\dots \qquad (3.61)$$

that is,

$$\langle 0|T\,\phi_1 \cdots \phi_n\, e^{i\int d^4x\, \mathcal{L}_{int}}|0\rangle$$

$$= \langle 0|T\,\phi_1 \cdots \phi_n|0\rangle + \langle 0|T\,\phi_1 \cdots \phi_n\, \mathcal{L}_{int}|0\rangle$$

$$+ \langle 0|T\,\phi_1 \cdots \phi_n \left(\mathcal{L}_{int}^2/2 \right)|0\rangle + \cdots \Big) \langle 0|T\, e^{i\int d^4x\, \mathcal{L}_{int}}|0\rangle \qquad (3.62)$$

!

It follows that

$$\frac{\langle 0|T\,\phi_1 \cdots \phi_n\, e^{i\int d^4x\, \mathcal{L}_{int}}|0\rangle}{\langle 0|T\, e^{i\int d^4x\, \mathcal{L}_{int}}|0\rangle} = \langle 0|T\,\phi_1 \cdots \phi_n\, e^{i\int d^4x\, \mathcal{L}_{int}}|0\rangle \qquad (3.63)$$

= all diagrams <u>without</u> vacuum bubbles

For (3.62) we use

$$\frac{1}{m!}\langle 0|T\phi_1\cdots\phi_n\,\mathcal{L}_{int}^{\,m}|0\rangle\Big|_{O(2\mathcal{L}_{int}\text{-contr.})}$$

$$= \frac{1}{m!}\langle 0|T\phi_1\cdots\phi_n\,\mathcal{L}_{int}^{\,2}|0\rangle\langle 0|T\mathcal{L}_{int}^{\,m-2}|0\rangle$$

$$\cdot\,\frac{m\cdot(m-1)}{2}$$

$$= \frac{1}{2}\langle 0|T\phi_1\cdots\phi_n\,\mathcal{L}_{int}^{\,2}|0\rangle\frac{1}{(m-2)!}\langle 0|T\mathcal{L}_{int}^{\,m-2}|0\rangle$$

In general the combinatorical factor for l-$\mathcal{L}_{int}$-contr. is

$$\frac{1}{m!}\binom{m}{l} = \frac{1}{m!}\,\frac{m!}{(m-l)!}\,\frac{1}{l!} = \frac{1}{(m-l)!}\,\frac{1}{l!}$$

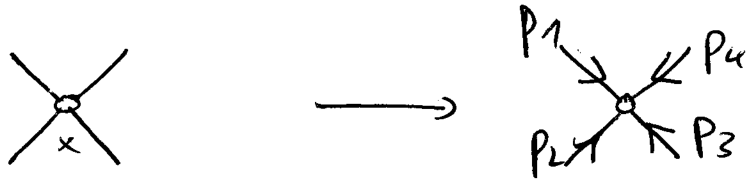
Most computations are done in momentum space: Fourier transforms

(i) $$\underset{p_1 \quad p_2}{D_F(x-y)} \longrightarrow \underset{\overset{\parallel}{D_F(p)}}{\frac{i}{p_1^{\,} - m^2 + i\varepsilon}} (2\pi)^4 \delta^4(p_1 - p_2)$$

(slight abuse of notation)

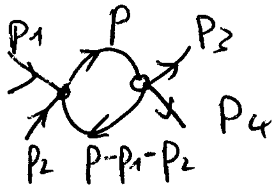
(ii) $$-i\lambda \int d^4x \; \Phi(x)^4 = -i\lambda \int \prod_{i=1}^{4} \frac{d^4 p_i}{(2\pi)^4} \; \Phi(p_i) \cdot (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)$$

$$-i\lambda \longrightarrow -i\lambda \, (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)$$



$p_4 = -(p_1 + p_2 + p_3)$

For example:

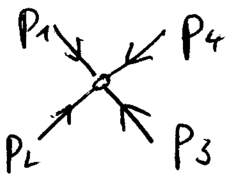


$$= (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \cdot \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\varepsilon} \frac{1}{(p - p_1 - p_2)^2 - m^2 + i\varepsilon}$$

Feynman rules in momentum space

(i) 

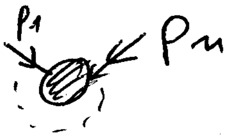
$= \dfrac{i}{p^2-m^2+i\varepsilon}$

(ii) 

$= -i\lambda$ and $p_4 = -(p_1+p_2+p_3)$

momentum conservat.

(iii) $\displaystyle\int \frac{d^4p}{(2\pi)^4}$ for each loop

(iv) $(2\pi)^4\, \delta^4\big(\sum_i p_i\big)$ for 

(v) multiplication with $1/S$

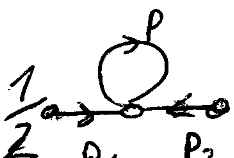
Examples:

(1) two-point fct.:

$$\langle T\,\phi(p_1)\phi(-p_2)\rangle = \frac{i}{p_1^2-m^2+i\varepsilon}\,(2\pi)^4\,\delta^4(p_1+p_2)$$

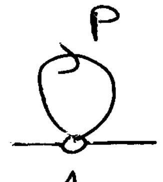
$\|$

$(2\pi)^4\,\delta^4(p_1+p_2)\cdot$ 

$+\,\frac{1}{2}$ 

$+\, O(\lambda^2)$

$1/S$



$$= -i\lambda \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} = -i\Pi + O(\lambda^2) \tag{3.64}$$

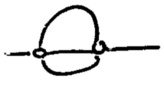
without ext. props.

Heuristics:

$$= \frac{i}{p^2-m^2+i\varepsilon} + \frac{i}{p^2-m^2+i\varepsilon}(-i\Pi)\frac{i}{p^2-m^2+i\varepsilon} + O(\lambda^2)$$

$$= \frac{i}{p^2-m^2-\Pi+i\varepsilon} + O(\lambda^2) \tag{3.65}$$

$\Rightarrow \quad m^2 - \Pi$ = interacting mass
finite!

in general (beyond 1-loop): e.g. 

$$\Pi \to \Pi(p)$$

Remarks: proper treatment: renormalisation

interacting vs free observables: LSZ-form.

Lehman, Symanzik, Zimmermann