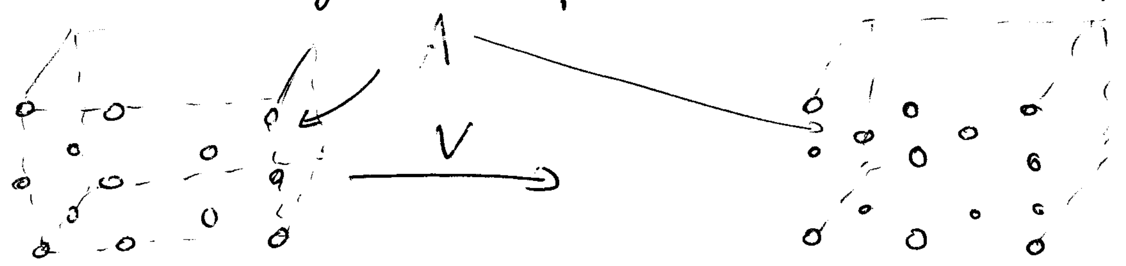
3.4. Cross Section

Fixed target experiment: Example



bunch of particles — bunch of particles

type B: bunch length l_B — type A: bunch length l_A

density ρ_B $\frac{\#}{\text{Area}\cdot\text{length}}$ — density ρ_A

velocity v — velocity 0

cross section σ:

$$\sigma = \frac{\text{Number of events} = N_{\text{events}}}{(N_B N_A)/A} \tag{3.66}$$

where

A scattering area (transverse)

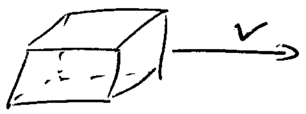
Space-dependent density:

$$(N_B\, N_A)/A = \int_A d^2x\ \rho_A(x)\, \rho_B(x)\, l_A l_B \qquad (3.67)$$

$$\Rightarrow \quad \sigma = \frac{N_{events}}{l_A l_B \int_A d^2x\ \rho_A(x)\, \rho_B(x)}$$

$$\text{const. densities} \rightarrow = \frac{N_{events}}{l_A \rho_A\, l_B \rho_B \cdot A} \qquad (3.68)$$

For the above example, we need to consider states, that are localised in space/mom.:



Wave packet: (see eq. (2.67), p. 26)

$$|f_{\vec{p}}\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}}\, f_{\vec{p}}(\vec{k})\, |\vec{k}\rangle \qquad (3.69)$$

with $f_{\vec{p}}(\vec{k})$ is peaked at $\vec{p}$

e.g. $f_{\vec{p}}(\vec{k}) \sim e^{-(\vec{k}-\vec{p})^2/\sigma^2}$ Gaußian

Normalisation: $\langle f_{\vec p} | f_{\vec p} \rangle = 1$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{1}{2\omega_{\vec k}} \frac{1}{2\omega_{\vec k}} f^*_{\vec p}(\vec k') f_{\vec p}(\vec k) \langle \vec k' | \vec k \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec k}} |f_{\vec p}|^2(\vec k) \qquad (3.70)$$

Gaußian is localised in $\vec k$ and $\vec x$: a

Fourier transform of a Gaußian is a Gaußian

In operator language,

$$|f_{\vec p}\rangle = \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} e^{i\vec k \vec x} f(\vec k)\, \phi(\vec x)|0\rangle$$

$$- i \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec k}} e^{ik\vec x} f(\vec k)\, \Pi(\vec x)|0\rangle$$

$$(3.71)$$

from

$$a^+(\vec k) = \int d^3x\, e^{i\vec k \vec x} \{ \omega_{\vec k}\, \phi(\vec x) - i\Pi(\vec x) \}$$

$$(3.72)$$

see p. 19-20

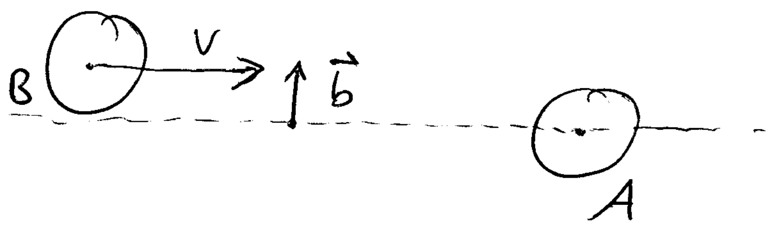
Remark: probab. for 1 part.: $\int \frac{d^3p}{(2\pi)^3} |\langle \vec q | f_{\vec p} \rangle|^2$

$$= \int \frac{d^3q}{(2\pi)^3} |f_{\vec p}(\vec q)|^2 = 1$$

Initial state in our case: $|\vec{k}_A \vec{k}_B\rangle = a^\dagger(\vec{k}_A)\, a^\dagger(\vec{k}_B)|0\rangle \cdot \sqrt{2\omega_{\vec{k}_A} 2\omega_{\vec{k}_B}}$

$$|i\rangle = \int \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_A}} \frac{1}{2\omega_{\vec{k}_B}} f_{\vec{p}_A}(\vec{k}_A)\, f_{\vec{p}_B}(\vec{k}_B) |\vec{k}_A \vec{k}_B\rangle \tag{3.73}$$

Impact parameter $\vec{b}$:



Translation is generated by momentum operator $\vec{P}$: (p. 31 eqs. (2.90), (2.91))

$$e^{-i\vec{P}\vec{b}}|\vec{k}\rangle = e^{-i\vec{k}\vec{b}}|\vec{k}\rangle \tag{3.74}$$

$$\Rightarrow e^{-i\vec{P}\vec{b}}|\delta_{\vec{p}}\rangle$$

$$= \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{i\vec{k}(\vec{x}-\vec{b})} \cdot \left(\phi(\vec{x})\,\omega_{\vec{k}} - i\pi(\vec{x})\right)|0\rangle \tag{3.75}$$

$\Rightarrow$ Initial state with impact parameter $\vec{b}$:

$$|i_{\vec{b}}\rangle = \int \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_A}} \frac{1}{2\omega_{\vec{k}_B}} f_{\vec{p}_A}(\vec{k}_A) f_{\vec{p}_B}(\vec{k}_B) \cdot e^{-i\vec{k}_B \vec{b}} |\vec{k}_A \vec{k}_B\rangle \tag{3.76}$$

Transition amplitude:

$$\langle \vec{p}_{f_1} \vec{p}_{f_2} | S | i_{\vec{b}} \rangle$$

with probability $|\langle \vec{p}_1 \vec{p}_2 | S | i_{\vec{b}}\rangle|^2$. The number of events in a dense beam is (single targ. part.: $N_A=1$)

$$N_{events} = \frac{N_B}{A} \cdot \int_A d^2b \, |\langle \vec{p}_{f_1} \vec{p}_{f_2} | S | i_b \rangle|^2 \tag{3.77}$$

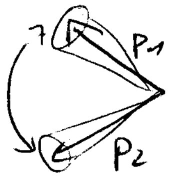
with cross section

$$\sigma(\vec{p}_1, \vec{p}_2) = \int d^2b \, |\langle \vec{p}_{f_1} \vec{p}_{f_2} | S | i_b \rangle|^2 \tag{3.78}$$

(3.66)

More realistic is a detection of a momentum region V_f, e.g.

'connected'



$$\Rightarrow \sigma(V_f) = \int_{V_f} \underbrace{\frac{d^3 p_{f1}}{(2\pi)^3} \frac{1}{2\omega_{\vec p_1}}}_{\frac{d^4 p_{f1}}{(2\pi)^4} 2\pi \delta(p_{f1}^2 - m^2) \leftarrow \text{on-shell!}} \frac{d^3 p_{f2}}{(2\pi)^3} \frac{1}{2\omega_{\vec p_2}} \int d^2 b \, |\langle \vec p_{f_1} \vec p_{f_2} | S | i_b \rangle|^2 \qquad (3.79)$$

or differentially and n particles

$$\boxed{d\sigma = \prod_{i=1}^{n} \frac{d^3 p_{fi}}{(2\pi)^3} \frac{1}{2\omega_{\vec p_i}} \int d^2 b \, |\langle \vec p_{f1} \cdots \vec p_{fn} | S | i_b \rangle|^2} \qquad (3.80)$$

Assume now that the $\vec p_i$ are not parallel to $\vec p_B$ (no (trivial) forward scattering).

We use that

$$S_{fi} = 1\!\!1_{fi} + i T_{fi}, \quad i T_{fi} = i \mathcal{M}_{fi} (2\pi)^4 \delta^4\left(\sum p_{fi} - \sum k_i\right)$$

and conclude

$$d\sigma = \prod_i \frac{d^3 p_{fi}}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}_i}} \int d^2 b \int \frac{d^3 k_A}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_A}} \cdot \frac{d^3 k_B}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_B}}$$

$$\cdot \int \frac{d^3 k_A'}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_A'}} \frac{d^3 k_B'}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_B'}} \; f_{\vec{p}_A}(\vec{k}_A) f_{\vec{p}_B}(\vec{k}_B) f^*_{\vec{p}_A}(\vec{k}_A') f^*_{\vec{p}_B}(\vec{k}_B')$$

$$\cdot e^{i\vec{b}(\vec{k}_B' - \vec{k}_B)} |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^4(\Sigma p_{fi} - \Sigma k_i)$$

$$\cdot (2\pi)^4 \delta^4(p_{fi} - \Sigma k_i') \quad (3.81)$$

with $k_1 = k_A$, $k_2 = k_B$.

Computation:

$$k_\perp = (k^1, k^2)$$

(i) $$\int d^2 b \, e^{i\vec{b}(\vec{k}_B' - \vec{k}_B)} = (2\pi)^2 \delta^2(k_{B_\perp}' - k_{B_\perp}) \quad (3.82)$$

(ii) $$\int d^3 k_A' \, d^3 k_B' \; \delta^4(\Sigma p_{fi} - \Sigma k_i') \, \delta^2(k_{B_\perp}' - k_{B_\perp})$$

$$= \int d k_A^{3'} \, d k_B^{3'} \; \delta(\Sigma p_{fi}^3 - \Sigma k_i'^3) \, \delta(\Sigma p_{fi}^0 - \Sigma k_i^{0'})$$

$$= \int d k_A^{3'} \; \delta(\Sigma p_{fi}^0 - \Sigma k_i^{0'}) \quad (3.83)$$

with $k_{B_\perp}' = k_{B_\perp}$, $k_{A_\perp}' = k_{A_\perp}$, $k_B^{3'} = \Sigma p_{fi}^3 - k_A^{3'}$

$$\Rightarrow \int d^3k_A' \, d^3k_B' \, \delta^4(\Sigma p_{fi} - \Sigma k_i') \, \delta^2(k'_{B_\perp} - k_{B_\perp})$$

$$= \int dk_A^{3'} \, \delta\Big(\Sigma p_{fi}^0 - \sqrt{\underbrace{\vec{k}_A'^2}_{k_{A_\perp}^2 + (k_A^{3'})^2} + m_A^2} - \sqrt{\underbrace{\vec{k}_B'^2}_{k_{B_\perp}^2 + (\Sigma p_i^3 - k_A^{3'})^2} + m_B^2}\Big)\Bigg|_{\substack{k_{A/B\perp} = k'_{A/B\perp} \\ \Sigma k_i^{3'} = \Sigma k_i^3 \\ \Sigma k_i^{0'} = \Sigma k_i^0}}$$

$$= \frac{1}{\left| k_A^{3'}/k_A^{0'} - k_B^{3'}/k_B^{0'} \right|} \xrightarrow{k_{A/B} \simeq p_{A/B}} \frac{1}{\underbrace{|v_A - v_B|}_{\text{rel. velocity in lab. frame}}} \qquad (3.84)$$

(iii) The wave packets $f_{\vec{p}_{A/B}}$ are localised about $\vec{p}_{A/B}$ hence we can substitute in all prefactors $k'_{A/B} \to p_{A/B}$.

It follows

$$d\sigma = \prod_i \frac{d^3 p_{fi}}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}_{fi}}} \frac{1}{4 p_A^0 p_B^0 |v_A - v_B|}$$

$$\cdot \int \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{1}{2k_A^0 \, 2k_B^0} |f_{\vec{p}_A}(\vec{k}_A)|^2 |f_{\vec{p}_B}(\vec{k}_B)|^2$$

$$\cdot |\mathcal{M}_{fi}|^2 \cdot (2\pi)^4 \delta^4(\Sigma p_{fi} - \Sigma k_i) \qquad (3.85)$$

Eq. (3.87) can be rewritten in a manifestly boost-invariant way:

$$d\sigma = \frac{1}{2\,w(s, m_A^2, m_B^2)} \cdot |\mathcal{M}_{fi}|^2 \, d\Pi_n \qquad (3.88)$$

with $w(x,y,z) = \sqrt{x^2+y^2+z^2-2xy-2xz-2yz}$

Exercise

Example: 2–2 scattering in ϕ^4-theory in the highly relativistic case

Phase space: $n=2$ in eq. (3.87), $p_{fi} = p_i$

$$\int d\Pi_2 = \int (2\pi)^4 \, \delta^4(p_1+p_2-(p_A+p_B))$$

$$\cdot \frac{d^3p_1}{(2\pi)^3}\frac{1}{2p_1^0}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2p_2^0} \qquad (3.89)$$

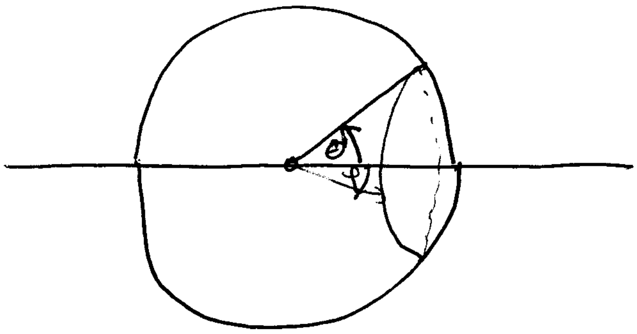
$$(p_A+p_B)^2 \gg m_A^2, m_B^2 \rightarrow \simeq \frac{1}{(2\pi)^2\, 4p_1^0p_2^0}\int d^3p_2\, \delta(p_1^0+p_2^0-\sqrt{s})$$

with comp. in CMS: $\vec{p}_1 = -\vec{p}_2 \Rightarrow p_1^0 = p_2^0$

We also use

$$d^3 p_2 = d\Omega \; |\vec{p}_2|^2 \, d|\vec{p}_2| \qquad (3.90)$$

with $d\Omega = d\varphi \; \sin\Theta \, d\Theta$



It follows, $p_1^0 + p_2^0 - \sqrt{s} \simeq 2p_2^0 - \sqrt{s} = 2|\vec{p}_2| - \sqrt{s}$, $p_i^0 = \sqrt{s}/2$

$$\int d\Pi_2 = \frac{1}{2} \frac{s/4}{(2\pi)^2 \underbrace{4\, p_1^0 p_2^0}_{4 \cdot s/4 = s}} \, d\Omega = \frac{1}{32\pi^2} d\Omega \qquad (3.91)$$

Now we use (see p. 45, eq. 3.23), that

$$|\mathcal{M}_{fi}|^2 = \lambda^2 \qquad (3.92)$$

and hence

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{2s} |\mathcal{M}_{fi}|^2 \int_{d\Omega(p_2)\ \text{fixed}} d\Pi_2 = \frac{\lambda^2}{64\pi^2 s}} \qquad (3.93)$$

Remarks: (i) dim. counting: λ^2/s, (ii) angle-indep.: spin 0

Finally: Computation of S-Matrix elements

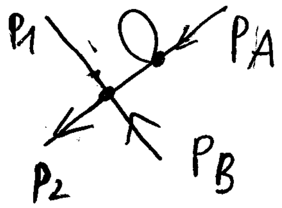
In the 2-2 scattering example we have used, that

$$|\mathcal{M}_{fi}|^2 = \lambda^2 + \mathcal{O}(\lambda^3) \qquad (3.94)$$

Expansion in Feynman diagrams:

$\langle \vec{p}_1\, \vec{p}_2 | iT | \vec{p}_A\, \vec{p}_B \rangle =$  $+ \Big($  + permuts$\Big)$

$+$  $+ \ldots \qquad (3.95)$

- Compute: 

with $\frac{1}{2}$  $= -i\pi$

$$= \frac{i}{p_A^2 - m^2 + i\varepsilon} \underbrace{(-i\pi)}_{}^{\mathcal{O}(\lambda)} (-i\lambda)$$

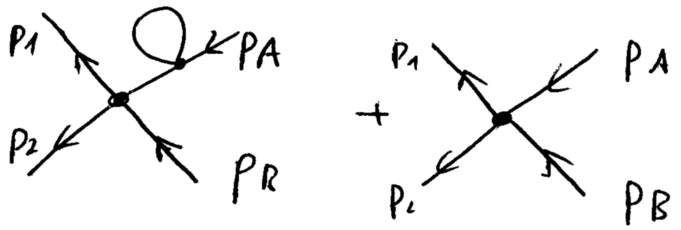
$$\cdot (2\pi)^4 \, \delta^4(p_A + p_B - p_1 - p_2)$$

$$(3.96)$$

$$= \left(\frac{i}{p_1^2 - m^2 + i\varepsilon} \right)^{-1} \frac{i}{(p_A^2 - m^2 + i\varepsilon)} (-i\Pi) \frac{i}{(p^2 - m^2 + i\varepsilon)} (-i\lambda)$$

$$\cdot\, \delta^4 (p_A + p_B - p_1 - p_2)$$

$\Rightarrow \frac{1}{2}$

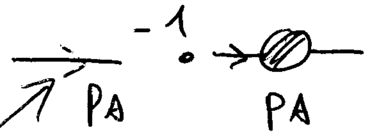


$$= -i\lambda \left(\frac{i}{p_1^2 - m^2 + i\varepsilon} \right)^{-1} \left[\frac{i}{p_A^2 - m^2 + i\varepsilon} + \frac{i}{p_A^2 - m^2 + i\varepsilon} (-i\Pi) \frac{i}{p_A^2 - m^2 + i\varepsilon} \right]$$

$$\cdot\, \delta^4 (p_A + p_B - p_1 - p_2)$$

$$= -i\lambda \left(\frac{i}{p_A^2 - m^2 + i\varepsilon} \right)^{-1} \frac{i}{p_A - (m^2 + \Pi) + i\varepsilon} \; \delta^4 (p_A + p_B - p_1 - p_2)$$

$$= \left[-i\lambda \, \delta^4 (p_A + p_B - p_1 - p_2) \right] \qquad (3.97)$$



(bare) free inverse propagation with p_A

full propagation with p_A

Remarks:

(i) $\left(\xrightarrow[p_A]{}\right)^{-1}$ related to the fact,

that the particle A in the initial state was prepared as a free state, only true for $t \to -\infty$. The correct in-state should relate to full (inverse) propagation

(ii) $\left(\xrightarrow[p_A]{}\right)^{-1} \to \left(\xrightarrow[p_A]{\bullet}\right)^{-1}$ leads to

$$1 = \left(\xrightarrow[p_A]{\bullet}\right)^{-1}\left(\xrightarrow[p_A]{\bullet}\right)$$ in eq. (3.97):

$\mathcal{M}_{fi}$ is computed by computing amputated, connected scattering diagrams.