3.5 LSZ - formalism

(Lehmann, Symanzik, Zimmermann)

We have seen that the naive preparation of our in-state lead to a factor

$—^{-1}\; —\!\circ\!— \neq 1$ in our scattering amplitudes.

We have encountered a similar problem with vacuum bubbles before.

We shall see that

$$\phi_H(t \to \mp\infty) \to Z^{1/2}\,\phi_{in/out} \quad \text{(weak op. equivalence)}$$

with $Z \leq 1$. So far, we have implicitly assumed $Z = 1$.

For determining Z, we carefully compute

$$\langle \Omega | \phi_H(x)\, \phi_H(y) | \Omega \rangle = \langle \phi(x)\, \phi(y) \rangle$$

$$= \sum_{\lambda, \vec{p}} \langle \Omega | \phi(x) | \lambda, \vec{p} \rangle_H \langle \lambda, \vec{p} | \phi(y) | \Omega \rangle \qquad (3.98)$$

boosts

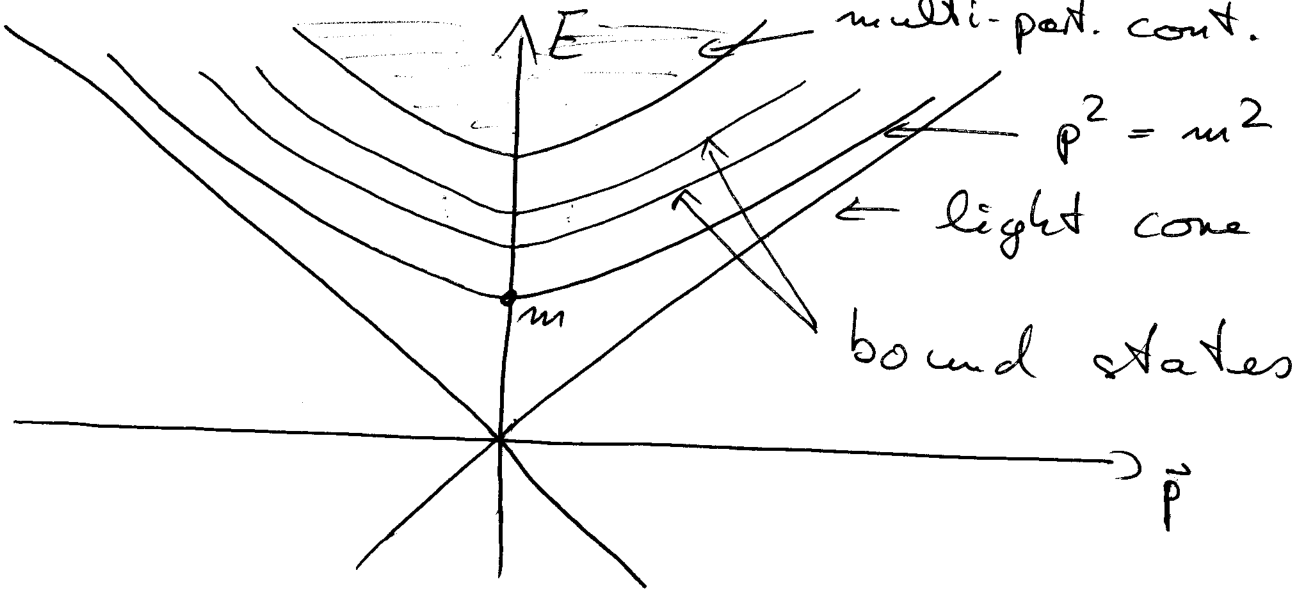
(i) Here $|\lambda,\vec{p}\rangle$ are the eigenstates of

H : $H|\lambda,\vec{p}\rangle = E_\lambda|\lambda,\vec{p}\rangle$

and $\vec{P}|\lambda,\vec{p}\rangle = \vec{p}_\lambda|\lambda,\vec{p}\rangle$

with $E_\lambda^2 - \vec{p}_\lambda^{\,2} = m_\lambda^2$ fixed. The states

$|\lambda,\vec{p}\rangle$ with fixed m_λ are connected by boosts.



(ii) $E_\lambda = 0$: $|\Omega\rangle$

(iii) $$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda\!\!\!\!\!\!\int \int \frac{d^3p_\lambda}{(2\pi)^3}\frac{1}{2E_\lambda(\vec{p})}|\lambda,\vec{p}\rangle$$

(iv) $\langle\Omega|\phi(x)|\Omega\rangle = 0$

We use that with $\hat{P} = (H, \vec{P})$ we have

$$\phi(x) = e^{i\hat{P}x}\,\phi(0)\,e^{-i\hat{P}x} \tag{3.99}$$

Then we get, $(x^0 \geq y^0)$

$$\langle \phi(x)\,\phi(y) \rangle = \sum\!\!\!\!\!\!\int_\lambda \overbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\lambda(\vec{p})}}^{\int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m_\lambda^2)\Theta(p^0_\lambda)} |\langle \Omega|\phi(0)|\lambda,\vec{p}\rangle|^2 \cdot e^{-ip_\lambda(x-y)}$$

$$= \sum\!\!\!\!\!\!\int_\lambda \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2-m_\lambda^2+i\varepsilon}\,e^{-ip(x-y)}$$

$$\cdot\, \underbrace{|\langle \Omega|\phi(0)|\lambda,\vec{p}\rangle|^2}_{|\langle \Omega|\phi(0)|\lambda\rangle|^2 \;\leftarrow\; |\lambda\rangle = |\lambda,0\rangle} \tag{3.100}$$

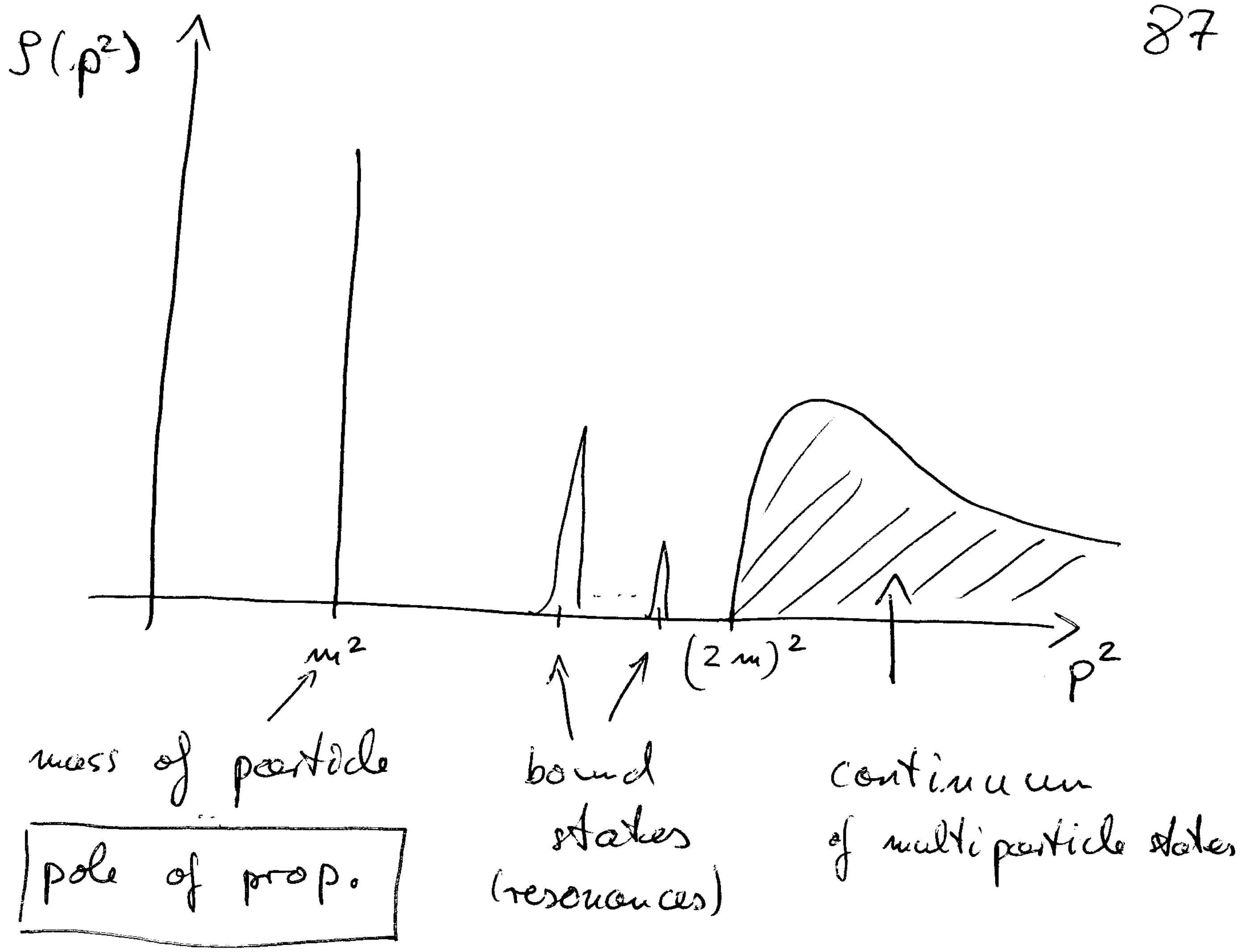
Boost-inv. $\longrightarrow$

In summary, by using the same steps for $x^0 \leq y^0$, (Källén - Lehmann)

$$\boxed{\langle T\,\phi(x)\,\phi(y)\rangle = \int_0^\infty \frac{d\mu^2}{2\pi}\,\rho(\mu^2)\,D_F(x-y;\mu^2)} \tag{3.101}$$

with spectral function

$$\rho(p^2) = \sum\!\!\!\!\!\!\int_\lambda (2\pi)\,\delta(p^2-m_\lambda^2)\,|\langle \Omega|\phi(0)|\lambda\rangle|^2 \tag{3.102}$$



ρ has the representation

$$\rho(p^2) = Z\, 2\pi\, \delta(p^2 - m^2) + \Theta(p^2 - \underset{\overset{\parallel}{m_1^2}}{m^2_{\text{first-res}}}) \cdots \tag{3.103}$$

and hence

$$\langle T\phi(x)\,\phi(y)\rangle = Z\, D_F(x-y;\, m^2) + \int_{m_1^2}^{\infty} \frac{dM^2}{(2\pi)}\, \rho(M^2)\, D_F(x-y;\, M^2) \tag{3.104}$$

carries one-particle pole of ϕ

Relation to ϕ_{in} on p. 84:

Consider one-particle states $|\lambda_1\rangle$ in $|\lambda\rangle\langle\lambda|$

$$\rho \sim \sum_{\substack{\text{one-particle}\\ \text{states } \lambda_1}} e^{-ip_\lambda(x-y)} |\langle\Omega|\phi(0)|\lambda_1\rangle|^2$$

with, $U = U(-\infty, 0)$,

$$|\langle\Omega|\phi(0)|\lambda_1\rangle|^2 = |\langle\Omega| U^{-1} U \phi U^{-1} U |\lambda_1\rangle|^2$$

$$= |{}_I\langle 0| Z^{1/2} \phi_{in}^{0} |\lambda_1\rangle_I|^2$$

$$= Z$$

Determination of Z:

Consider now $\langle \phi(x)\,\phi(y)\rangle$, no time-ord.!

Then D_F in (3.104) is substituted by D.

Note also that

$$\left[\frac{\partial}{\partial y^0}\langle[\phi(x),\phi(y)]\rangle\right]_{x^0=y^0} = \langle[\phi(x),\,\Pi(y)]_{x^0=y^0}\rangle$$

$$= i\,\delta^3(\vec{x}-\vec{y}) \qquad (3.105)$$

and $$\left[\frac{\partial}{\partial y^0}\big(D(x-y)-D(y-x)\big)\right]_{x^0=y^0} = i\,\delta^3(\vec{x}-\vec{y})$$

Evaluate $\int d^3x \langle[\phi(x),\Pi(y)]_{x^0=y^0}$ with eqs. (3.104), (3.105)

$$\boxed{1 = Z + \int_{m_1^2}^{\infty}\frac{d\mu^2}{(2\pi)}\,\underset{\geq 0}{\rho(\mu^2)}} \qquad (3.106)$$

Eq. (3.106) entails

$$\boxed{0 \leq Z \leq 1} \qquad (3.107)$$

Remarks:

(i) $1-Z$ accounts for overlap of $\phi|\Omega\rangle$ with multi-particle states

(ii) $Z=1$ in free theory

$Z<1$ in interacting theory

(iii) limit $t\to\mp\infty$:

$$\phi(x) \longrightarrow Z^{1/2}\phi_{in/out} \quad \text{weak op.}$$

(iv) Propagator on-shell:

$$D_F(p^2\to m^2) \simeq \frac{iZ}{p^2-m^2+i\varepsilon} \qquad (3.108)$$

$$\left(= \int d^4x\, e^{ipx} \langle T\phi(x)\,\phi(0)\rangle \right)$$

Note: m^2 not simply mass-parameter m_0^2 in Lagrangian

LSZ-reduction formula:

We now extend the analysis of the two-point function to an n-point function. The latter will be related to S-matrix elements.

As in eq. (3.108) we evaluate the Fourier transform

$$\int d^4x\, e^{ipx} \langle T\, \phi(x)\, \phi(x_2) \cdots \phi(x_n) \rangle \tag{3.109}$$

With $T_+ > x_2^0, \ldots, x_n^0$, $T_- < x_2^0, \ldots, x_n^0$ we split

$$\int dx^0\, e^{ip^0x^0} = \left(\int_{-\infty}^{T_-} + \int_{T_-}^{T_+} + \int_{T_+}^{\infty} \right) dx^0\, e^{ip^0x^0}$$

finite

p on-shell: $p^2 = m^2$

pole

It follows

$$\int d^4x\, e^{ipx} \langle T\, \phi(x)\, \phi_2 \cdots \phi_n \rangle$$

$$= \int_{T_+}^{\infty} d^4x\, e^{ipx} \langle \phi(x)\, T \phi_2 \cdots \phi_n \rangle + \left(\int_{T_-}^{T_+} + \int_{-\infty}^{T_-} \right) \cdots$$

$$= \int_{T_+}^{\infty} d^4x\, e^{ipx} \sum\!\!\!\!\!\!\int_{\lambda} \int \frac{d^3q}{2\pi^3} \frac{1}{2\omega_{\vec{q}}} \langle \phi(x) | \lambda, \vec{q} \rangle \langle \lambda, \vec{q} | T \phi_2 \cdots \phi_n \rangle$$

$$+ \cdots \qquad (3.110)$$

We use $\langle \phi(x) | \lambda, \vec{q} \rangle = \langle \Omega | \phi(0) | \lambda \rangle e^{-iqx}$:

$$\sum\!\!\!\!\!\!\int_{\lambda} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_{\vec{q}}} dx_0\, e^{i(p^0 - q^0 + i\varepsilon)x^0} \langle \Omega | \phi(0) | \lambda, \vec{p} \rangle \langle \lambda, \vec{p} | T \phi_2 \cdots \phi_n \rangle \cdot \underbrace{(2\pi)^3 \delta^3(\vec{p} - \vec{q})}_{\int d^3x\, e^{-i(\vec{p}-\vec{q})\vec{x}}}$$

$$= \sum\!\!\!\!\!\!\int_{\lambda} \frac{1}{2\omega_{\vec{p}}} \frac{i\, e^{i(p^0 - \omega_{\vec{p}} + i\varepsilon)T_+}}{p^0 - \omega_{\vec{p}} + i\varepsilon} \langle \Omega | \phi(0) | \lambda, \vec{p} \rangle \langle \lambda, \vec{p} | \phi_2 \cdots \phi_n \rangle \qquad (3.111)$$

For $p^0 \to \omega_{\vec{p}}$: (using Kallén-Lehmann)

$$\lim_{p^0 \to \omega_{\vec{p}}} \int_{T_+}^{\infty} d^4x\, e^{ipx} \langle T \phi(x) \cdots \phi(x_n) \rangle$$

$$= \frac{i Z^{1/2}}{p^2 - m^2 + i\varepsilon} \langle \lambda, \vec{p} | T \phi_2 \cdots \phi_n \rangle$$

$$+ \text{finite} \qquad (3.112)$$

Evaluation of $\int_{-\infty}^{T_-}$ - term:

$$\lim_{p^0 \to -\omega_{\vec{p}}} \int_{-\infty}^{T_-} d^4x \, e^{ipx} \langle (T \phi_2 \cdots \phi_n) \phi(x) \rangle$$

$$= \frac{i Z^{1/2}}{p^2 - m^2 + i\varepsilon} \langle T \phi_2 \cdots \phi_n | -\vec{p} \rangle \qquad (3.113)$$

$$+ \text{finite}$$

The last term $\int_{T_-}^{T_+} \cdots$ is finite as the integration interval has a finite length (compact)

Remarks:

(i) The above analysis can be repeated iteratively for all $\phi(x_i)$

(ii) Strictly speaking one should separate the fields spacially:

$$\int d^4x \, e^{ipx} \to \int \frac{d^3k}{(2\pi)^3} e^{ipx} f_{\vec{p}}(\vec{k})$$

(iii) states $|\vec{p}\rangle$ are at time $t \to -\infty$

states $\langle \vec{p}|$ are at time $t \to +\infty$

and

$$
{}_{-\infty}\langle \vec{p}_1 \cdots \vec{p}_n | \vec{k} \cdots \vec{k}_m \rangle_{+\infty}
$$

$$
= \langle \vec{p}_1 \cdots \vec{p}_n | S | \vec{k}_1 \cdots \vec{k}_m \rangle \qquad (3.114)
$$

$\Rightarrow$ LSZ Reduction formula:

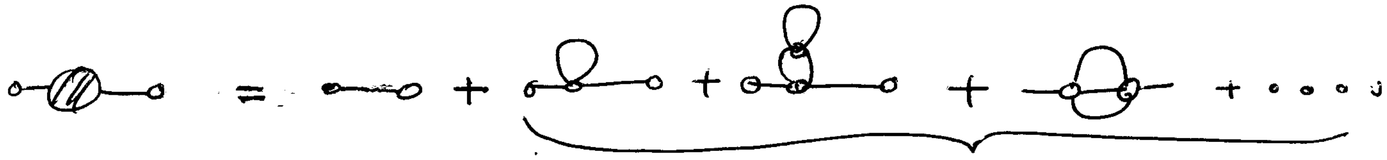
$$
\boxed{\begin{aligned}
&\langle \vec{p}_1 \cdots \vec{p}_n | S | \vec{k}_1 \cdots \vec{k}_m \rangle \Big|_{\text{on-shell}} \\
&= \int \prod_{i=1}^{n} d^4 x_i \, e^{i p_i x_i} \prod_{j=1}^{m} d^4 y_j \, e^{-i k_j y_j} \\
&\cdot \prod_{i=1}^{n} \left(\partial_{x_i}^2 + m^2 \right) \prod_{j=1}^{m} \left(\partial_{y_j}^2 + m^2 \right) \qquad (3.115) \\
&\cdot \left[\frac{i}{Z^{1/2}} \right]^{n+m} \langle T \phi(x_1) \cdots \phi(x_n) \phi(y_1) \cdots \phi(y_m) \rangle
\end{aligned}}
$$

on-shell: $p^2 = m^2 \leftarrow$ phys. mass pole

$m_0^2 \leftarrow$ parameter in Lagrangian

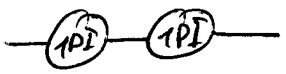
Structure of $\langle T \phi_1 \dots \phi_n \rangle$:

Example $n=2$:



$\Pi(p)$:  1PI : one-particle irreducible,

cannot be split by cutting <u>one</u> line

$+$  $+ \dots$

$$= \frac{i}{p^2 - m_0^2 + i\varepsilon} + \frac{i}{p^2 - m_0^2 + i\varepsilon} \left(-i\Pi(p) \right) \frac{i}{p^2 - m_0^2 + i\varepsilon}$$

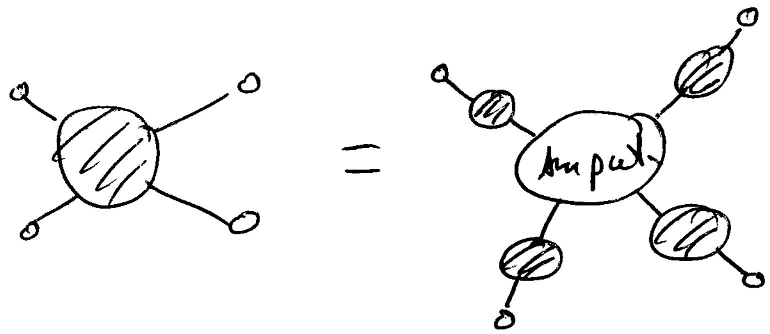
$$+ \dots$$

$$= \frac{i}{p^2 - \left(m_0^2 + \Pi(p) \right) + i\varepsilon} \qquad (3.116)$$

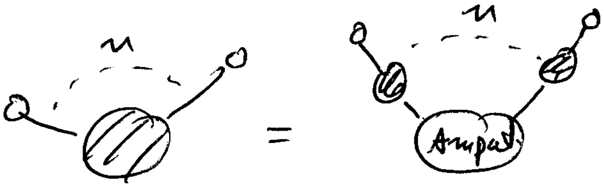
with

$$\frac{i}{p^2 - \left(m_0^2 + \Pi(p) \right) + i\varepsilon} \xrightarrow{p^2 \to m^2} \frac{iZ}{p^2 - m^2 + i\varepsilon} \qquad (3.117)$$

$n = 4$:

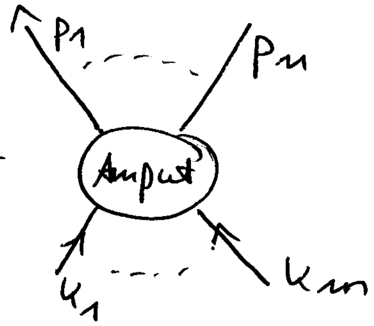


in general:



This entails for S-matrix elements with (3.115):

$$\boxed{\langle \vec{p}_1 \ldots \vec{p}_n | S | \vec{k}_1 \ldots \vec{k}_m \rangle \big|_{\text{on-shell}} = Z^{(n+m)/2}}$$



Remarks:

(i) Z is called wave function (or field strength) renormalisation, as it multiplies the field.

Note that

$$\langle T Z^{1/2}\phi(x)\, Z^{-1/2}\phi(y)\rangle_{p^2\to m^2} = D_F(x-y; m^2)$$

Z re-normalises the field

(ii) With (i) we see that

$$Z^{(n+m)/2}\,\langle T\,\phi(p_1)\cdots\phi(k_m)\rangle_{\text{amp.}}$$

$$\simeq Z^{(n+m)/2}\prod_i \frac{p_i^2-m^2}{iZ}\prod_j \frac{k_j^2-m^2}{iZ}\langle T\,\phi(p_1)\cdots\phi(k_m)\rangle$$

$$= \prod_i \frac{(p_i^2-m^2)}{i}\prod_j \frac{(k_j^2-m^2)}{i}\underbrace{\langle T\, Z^{-1/2}\phi(p_1)\cdots Z^{-1/2}\phi(k_m)\rangle}_{\text{expect. value of re-normalised fields}}$$