4. Fermions

4.1 Fields and Lorentz invariance

So far we have discussed the quantisation of a scalar field: (spin 0)

$$\phi(x) \xrightarrow[\text{trafo}]{\text{Lorentz}} \phi'(x') = \phi(x) \qquad (4.1)$$

$$\text{with } x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad (4.2)$$

$$\left(\text{and } \phi'(x) = \phi(\Lambda^{-1} \circ x) \right)$$

Vector fields: (spin 1)

$$\begin{aligned} A^\mu(x) &\longrightarrow \Lambda^\mu{}_\nu A^\nu(x) \\ A'^\mu(x) &= \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1} \circ x) \end{aligned} \qquad (4.3)$$

Tensor fields:

$$F^{\mu\nu}(x) \longrightarrow \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}(x) \qquad (4.4)$$

In general:

representation

$$\phi^i(x) \to \mathcal{R}(\Lambda)^i{}_j \, \phi^j(x) \qquad (4.5)$$

with general index i, e.g. $i = \{\}, \nu, \nu\nu, \ldots$

scalar: $R(\Lambda) = 1$ trivial rep

vector: $R(\Lambda) = \Lambda$ fundamental rep

(2nd rank) tensor: $R(\Lambda) = (\Lambda^\nu{}_\rho \Lambda^\nu{}_\sigma)$ tensor rep

Representation $\mathcal{R}: G \to \mathcal{R}(G)$, G group:

$$\mathcal{R}(\mathbb{1}) = \mathbb{1} \; ; \quad R(g \cdot h) = \mathcal{R}(g) \circ \mathcal{R}(h) \qquad (4.6)$$

Example: $SO(3)$ rotations in $\mathbb{R}^3$

(i) trivial rep.: $\mathcal{R}(\Lambda) = 1$ $\quad \Lambda \in SO(3)$

(ii) fundamental rep.: $R(\Lambda) = \Lambda$ $\quad$ Lie group

Lie group: $\Lambda = e^{i\vec{\omega}\vec{J}}$ with J^1, J^2, J^3

Lie algebra

J^i: generators

(4.7)

J^i generate infinitesimal rotations with axis is x^i: $(\partial^i x^j = -\delta^i{}_j)$

$$J^i = -i\varepsilon^{ijk}\, x^j \partial^k \tag{4.8}$$
$$= -\frac{1}{2}\varepsilon^{ijk} J^{jk} \quad , \quad J^{jk} = -i(x^j\partial^k - x^k\partial^j)$$

Lie-algebra:

$$[J^i, J^j] = i\varepsilon^{ijk} J^k \quad \text{check!} \tag{4.9}$$

QM: n-dim repres. of spin s with $n=2s+1$

Spin 1/2: generators $\sigma^i_{1/2}$ with

$$[\sigma^i_{1/2}, \sigma^j_{1/2}] = i\varepsilon^{ijk}\sigma^k_{1/2} \tag{4.10}$$

with Pauli matrices σ:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

spinor representation (4.11)

Remarks:

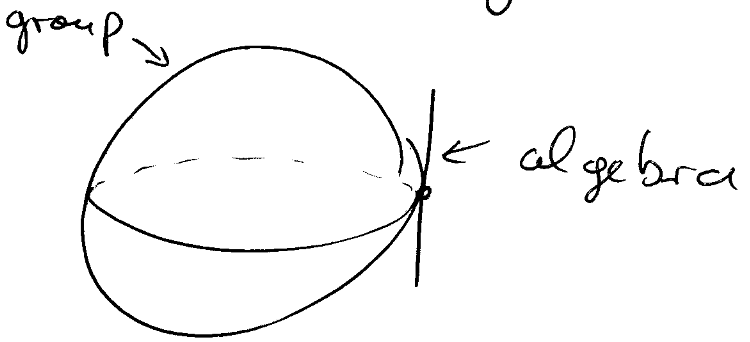
(i) The Lie algebra provides local information about the Lie group (tangential space). Important examples:

(1) $SO(3)$ and $SU(2) \simeq S^3$

Lie algebra:

$$[t^a, t^b] = i \underbrace{\varepsilon^{abc}}_{\text{structure constant}} t^c$$

generators



(Lie group = diff. manifold)

$SU(2)$ is universal double covering of $SO(3) \simeq \mathbb{R}P^3$

(2) $SO(1,3)$ and $SL(2,\mathbb{C})$

$\Downarrow$

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + i\,T_\mu{}^\nu \quad \text{infinitesimally} \tag{4.14}$$

$$SO(1,3):\ \Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta_{\nu\sigma} = \eta_{\mu\rho}$$

$$\Rightarrow (\delta_\mu{}^\nu + i\,T_\mu{}^\nu)(\delta_\rho{}^\sigma + i\,T_\rho{}^\sigma)\eta_{\nu\sigma} = \eta_{\mu\sigma} + O(T^2)$$

$$\Rightarrow \boxed{T_{\mu\rho} + T_{\rho\mu} = 0} \tag{4.15}$$

We conclude that T has $\frac{16 - \overset{\text{diag}}{4}}{\underset{\text{antisym}}{2}} = 6$ indep. var.

Generators M:

$$\boxed{T_\mu{}^\nu = \frac{\omega^{\rho\sigma}}{2}(M_{\rho\sigma})_\mu{}^\nu} \tag{4.16}$$

with Lie algebra

$$\boxed{[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho})} \tag{4.17}$$

Eq. (4.17) is the Lie algebra of $SO(1,3)$ rotations. To see this, we extend the $SO(3)$-generators of rotations, J^{ij} in eq. (4.8) to boosts (J^{0i}):

$$J^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \qquad (4.18)$$

The $J^{\mu\nu}$ satisfy (4.17).

General representations: look for $\mathcal{M}$ that satisfy (4.17).

E.g. fundamental representation:

$$\left(\mathcal{M}^{\mu\nu}\right)_{\rho\sigma} = i\left(\delta^\mu{}_\rho \delta^\nu{}_\sigma - \delta^\mu{}_\sigma \delta^\nu{}_\rho\right) \qquad (4.19)$$

Boosts and rotations are given by

$$J_i = \tfrac{1}{2}\varepsilon_{ijk}\mathcal{M}_{jk} \quad \text{rotations}$$

$$K_i = \mathcal{M}_{0i} \quad \text{boosts} \qquad (4.20)$$

Example for boost: (along x_1-axis)

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & 0 \\ & & 1 & \\ & 0 & & 1 \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$= \left(e^{\omega K_1}\right)^\mu{}_\nu$$ with rapidity ω

where

$$\omega = \operatorname{artanh} v/c$$

and generator

$$K_1 = \begin{pmatrix} 0 & -1 & & 0 \\ -1 & 0 & & \\ & 0 & & 0 \end{pmatrix}$$

The generators in eq. (4.20) make the structure of the Lorentz group apparent:

$$[J^i, J^j] = i\varepsilon^{ijk} J^k$$

$$[K^i, K^j] = -i\varepsilon^{ijk} J^k \qquad (4.21)$$

$$[J^i, K^j] = i\varepsilon^{ijk} K^k$$

(ii)* $SU(2)$ and $SL(2,\mathbb{C})$ with generators $(J^i + iK^i, J^i - iK^i)$ are universal covering groups of $SO(3)$ and $SO(1,3)$ respectively.

universal covering group $\tilde{G}$ of G:
'simply connected group $\tilde{G} \supseteq G$'