4.3 Quantisation

First we try to quantise fermions similarily to scalars (bosons).
The general solution to the Dirac-equation follows as

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p^0}} \sum_s \left\{ e^{-ipx} a_s(\vec{p}) \cdot u_s(\vec{p}) + e^{ipx} b_s^+(\vec{p})\, v_s(\vec{p}) \right\} \tag{4.68}$$

Question: What are the properties of the creation/annihilation operators a, b / a^+, b^+?

(i) Hamiltonian (see eq. (4.50), p. 112)

$$H = \int d^3x\, \mathcal{H} = \int d^3x\, \underbrace{\psi^+(\vec{x})\gamma^0}_{-i\pi_4}(i\vec{\gamma}\vec{\partial} + m)\psi(\vec{x})$$

p. 118a (iv)

$$= \int \frac{d^3p}{(2\pi)^3} \frac{2p^0}{2p^0} p^0 \sum_s \left\{ a_s^+(\vec{p})\, a_s(\vec{p}) - b_s(\vec{p})\, b_s^+(\vec{p}) \right\} \qquad (4.69)$$

Norm

where we have used (4.66) and

$$(\vec{\gamma}\vec{p} + m)\, u(p) = [-(\not{p} - m) + \gamma^0 p^0]\, u(p)$$

$$= \gamma^0 p^0 u(p) \qquad (4.70)$$

and $(-\vec{\gamma}\vec{p} + m)\, v(p) = -\gamma^0 p^0 v(p)$.

minus in (4.69)

Using commuting operators, e.g.

$$b_s b_s^+ = b_s^+ b_s + \text{c-number}$$

$$\Rightarrow H \simeq \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\, p_0 \sum_s \left\{ a_s^+(\vec{p})\, a_s(\vec{p}) - b_s^+(\vec{p})\, b_s(\vec{p}) \right\} \qquad (4.71)$$

↯

$\Rightarrow$ suggests the use of $b_s b_s^+ = -b_s^+ b_s + \text{c-number}$

(ii) Demanding

$$[\psi(\vec{x}), i\psi^\dagger(\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{4.72}$$

implies

$$[a_s(\vec{p}), a_r^\dagger(\vec{q})] = (2\pi)^3\delta(\vec{p}-\vec{q})\,\delta_{rs}$$

$$= \underset{!}{-}[b_s(\vec{p}), b_r^\dagger(\vec{q})]$$

rescues causality

but does not cure (4.71)!

We conclude

$$\boxed{\begin{aligned} \{a_s(\vec{p}), a_r^\dagger(\vec{q})\} &= (2\pi)^3\delta(\vec{p}-\vec{q}) \\ \{b_s(\vec{p}), b_r^\dagger(\vec{q})\} &= (2\pi)^3\delta(\vec{p}-\vec{q}) \end{aligned}} \tag{4.73}$$

b-b, a-a, b-$a^{(\dagger)}$ anticommutators vanish,

in particular $a_s(\vec{p})a_s(\vec{p}) = 0$

a Grassmann variable

and hence

$$\boxed{\begin{aligned}\{\psi_\xi(\vec{x}),\ \psi^\dagger_{\xi'}(\vec{y})\} &= \delta_{\xi\xi'}\delta^3(\vec{x}-\vec{y})\\ \{\psi_\xi(\vec{x}),\ \psi_{\xi'}(\vec{y})\} = 0 &= \{\psi^\dagger_\xi(\vec{x}),\ \psi^\dagger_{\xi'}(\vec{y})\}\end{aligned}} \qquad (4.74)$$

(iii) Fock space

Vacuum $|0\rangle$: $a_s(\vec{p})|0\rangle = b_s(\vec{p})|0\rangle = 0$

1 Particle states: $|\vec{p},s\rangle = \sqrt{2\omega_{\vec{p}}}\, a^\dagger_s(\vec{p})|0\rangle$

anti-particle: $\sqrt{2\omega_{\vec{p}}}\, b^\dagger_s(\vec{p})|0\rangle$ $\qquad (4.75)$

2 particle states: $\sim a^\dagger_s(\vec{p})\, a^\dagger_r(\vec{q})|0\rangle$

$$= \underset{!}{-}\, a^\dagger_r(\vec{q})\, a^\dagger_s(\vec{p})|0\rangle \qquad (4.76)$$

$\boxed{\text{States are anti-symmetric}}$

in particular:

$$\boxed{a^\dagger_r(\vec{p})\, a^\dagger_r(\vec{p})|0\rangle = 0} \qquad (4.77)$$

Normalisation: $\langle\vec{q},r|\vec{p},s\rangle = 2\omega_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q})\,\delta_{rs}$

(iv) Cont. symmetries

Four-momentum:

$$P^0 = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, p^0 \sum_s \left\{ a_s^+(\vec{p})\, a_s(\vec{p}) + b_s^+(\vec{p})\, b_s(\vec{p}) \right\} \qquad (4.78a)$$

$$= H$$

Fermions, Anti-fermions with $E = p^0 > 0$

$$P^i = \int \frac{d^3p}{(2\pi)^3}\, p^i \sum_s \left\{ a_s^+(\vec{p})\, a_s(\vec{p}) + b_s^+(\vec{p})\, b_s(\vec{p}) \right\} \qquad (4.78b)$$

Ψ is a complex field, the Lagrangian is invariant under $\Psi \to e^{ie\alpha}\Psi$, $\bar{\Psi} \to \bar{\Psi} e^{-ie\alpha}$

see p. 116, eq. (4.56)

Noether charge: $Q = \int d^3x\, j^0 = e\int d^3x\, \Psi^+(x)\Psi(x)$

$$\Rightarrow \boxed{Q \simeq e \int \frac{d^3p}{(2\pi)^3} \sum_s \left\{ a_s^+(\vec{p})\, a_s(\vec{p}) - b_s^+(\vec{p})\, b_s(\vec{p}) \right\}} \qquad (4.79)$$

e: elementary charge

$a_s^+(\vec{p})\, a_s(\vec{p})$: Fermion with charge e

$b_s^+(\vec{p})\, b_s(\vec{p})$: Anti-Fermion with charge $-e$

(v) Propagator:

$$\langle 0|\, \psi_\xi(x)\, \bar\psi_{\xi'}(y)\,|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\sum_s (u_s)_\xi (\bar u_s)_{\xi'}\, e^{-ip(x-y)} \qquad (4.80a)$$

eq. (4.66) $\longrightarrow$ $$= (i\partial\!\!\!/_x + m)_{\xi\xi'} \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0} e^{-ip(x-y)}}$$

and

$$\langle 0|\, \bar\psi_{\xi'}(y)\psi_\xi(x)\,|0\rangle \qquad \text{Scalar prop.}$$

$$= -(i\partial\!\!\!/_x + m)_{\xi\xi'} \overbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0} e^{-ip(y-x)}} \qquad (4.80b)$$

global minus sign

$\Rightarrow$ Feynman propagator:

$$S_{F_{\xi\xi'}}(x-y) = \langle 0|\, T\,\psi_\xi(x)\,\bar\psi_{\xi'}(y)\,|0\rangle \qquad (4.81)$$

$$= \int \frac{d^4p}{(2\pi)^4}\, \frac{i(p\!\!\!/ + m)_{\xi\xi'}}{p^2-m^2+i\varepsilon}\, e^{-ip(x-y)}$$

with time-ordering

$$T\,\psi(x)\bar\psi(y) = \Theta(x^0-y^0)\,\psi(x)\,\bar\psi(y) - \Theta(y^0-x^0)\,\bar\psi(y)\,\psi(x) \qquad (4.82)$$

Feynman rules:

We can directly take over the results for the scalar theory, chapter 3.2 – 3.5, but we have to take care of the anti-symmetry of fermions. We have already introduced

$$T\psi\bar{\psi} = -T\bar{\psi}\psi$$

Accordingly, if we define contractions as in the scalar theory, it follows

$$\begin{aligned}\psi(x)\bar{\psi}(y) &= \langle 0|T\,\psi(x)\,\bar{\psi}(y)|0\rangle \\ &= S_F(x-y) \qquad (4.83) \\ &= -\bar{\psi}(y)\,\psi(x)\end{aligned}$$

Furthermore

$$\ldots\psi\;\psi^n\;\bar{\psi}^m\;\bar{\psi}\ldots = (-1)^{n+m}\ldots\psi\bar{\psi}\;\psi^n\;\bar{\psi}^m \qquad (4.84)$$

Also $\quad :a\,a^\dagger: \;=\; -:a^\dagger a: \;=\; a^\dagger a \qquad (4.85)$

$$\Rightarrow\; :\psi_1 \cdots \psi_n \overset{(-)}{\psi}_{n+1} \cdots :$$

$$= -:\psi_1 \cdots \overset{(-)}{\psi}_{n+1}\, \psi_n \cdots :$$

$\Rightarrow$ Wick's theorem: (eq. (3.52)) $\qquad (4.86)$

$$\boxed{\begin{aligned} &T\; \psi(x_1)\cdots\psi(x_n)\,\bar\psi(x_{n+1})\cdots\bar\psi(x_{n+m}) \\ &\quad = :\psi(x_1)\cdots\bar\psi(x_{n+m}) + \text{all contractions}: \qquad (4.87)\end{aligned}}$$

Simpl(est) interacting theory:

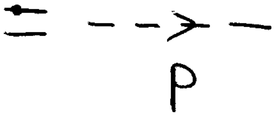
$$\mathcal{L} = \mathcal{L}_{scalar} + \mathcal{L}_{Dirac} + \mathcal{L}_I \qquad (4.88)$$

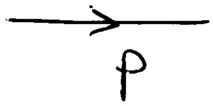
with

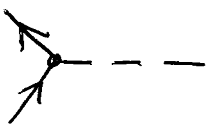
$$\mathcal{L}_I = -h\,\bar\psi\,\phi\,\psi \qquad (4.89)$$

Yukawa theory

Propagators:

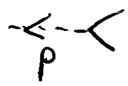
(a) ϕ: $\quad \overbrace{\phi \quad \phi}$ $\doteq$  $= \dfrac{i}{p^2 - m_\phi^2 + i\varepsilon} \qquad (4.90)$

(b) ψ: $\quad \overbrace{\psi \quad \bar\psi}$ $=$  $= \dfrac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\varepsilon} \qquad (4.91)$

Vertex:  $= -ig h$

External leg contraction

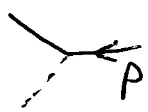
(a) $\quad \phi|\vec{p}\rangle := 1 \quad =: \langle\vec{p}|\phi$

(b) $$\underbrace{\psi(x)|}_{\text{annihil.}}|\vec{p},s\rangle = \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{2p^0}}{\sqrt{2q^0}} \sum_r e^{-iqx} a_r(\vec{q})\, a_s^\dagger(\vec{p})|0\rangle$$

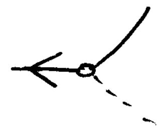
$$= \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0} \sum_r e^{-iqx} \underbrace{\{a_r(\vec{q}), a_s^\dagger(\vec{p})\}}_{\sim\delta(\vec{q}-\vec{p})}|0\rangle$$

$$= e^{-ipx} u_s(\vec{p}) \qquad (4.92)$$

$\Rightarrow \boxed{\psi|\vec{p},s\rangle := u_s(\vec{p})} =$  $\qquad (4.93)$

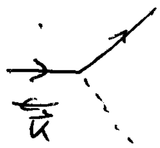
Analogously:

$$\langle \vec p, s | \bar\psi := \bar u_s(\vec p) = \quad \text{(4.94)}$$



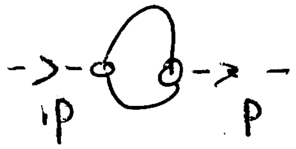
Anti-fermions:

$$= \bar\psi \underset{\substack{\| \\ b^+_s(\vec k)|0\rangle}}{|\vec k, s\rangle} := \bar v^s(\vec k) \quad \text{(4.95)}$$



$$\langle \vec k, s | \psi := v^s(\vec k) \quad \text{(4.96)}$$

Loops: e.g.



$$\Rightarrow \langle \vec q \,|\, (\phi\, \bar\psi\, \psi)(\phi\, \bar\psi\, \psi) \,|\, \vec p \rangle$$
$$\sim - \langle \vec q | \phi \,(\psi \bar\psi\ \psi \bar\psi)\, \phi | \vec p \rangle \quad \text{(4.97)}$$

closed fermionic Loops lead to minus signs

loop integral & Dirac traces on p. 128a

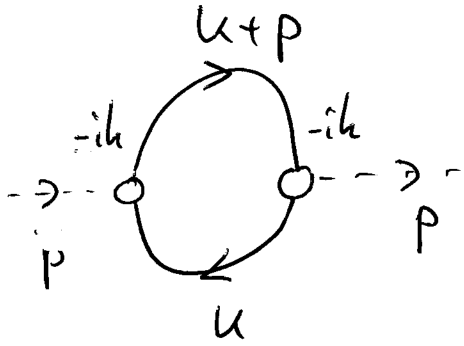
$$\Rightarrow \quad -(-ih)^2 \int d^4x \int d^4y \, \langle \vec{q} \,|\, \phi_x \, \psi_{x\,\xi} \, \bar{\psi}_{y\,\eta} \, \psi_{y\,\eta} \, \bar{\psi}_{x\,\xi} \, \phi_y \,|\, \vec{p} \rangle$$

trace

$$\simeq \; + h^2 \int \frac{d^4k}{(2\pi)^4} \, \mathrm{tr}\left(\frac{\not{k} + m_\psi}{p^2 - m_\psi^2} \; \frac{\not{k} + \not{p} + m_\psi}{(k+p)^2 - m_\psi^2} \right)$$

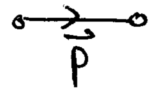
i^2 from props.

from

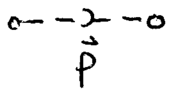


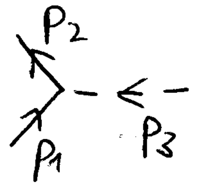
In summary: (see p. 68)

Feynman rules in momentum space

(i)  $= \dfrac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\varepsilon}$

(4.98)

 $= \dfrac{i}{p^2 - m_\phi^2 + i\varepsilon}$

(ii)  $= -ih$ and $p_1 + p_3 = p_2$ (4.99)

(iii) $\int \frac{d^4 p}{(2\pi)^4}$ for each loop

(−) for each fermion loop

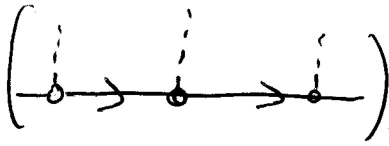
(iv) $(2\pi)^4 \, \delta^4(\sum_i p_i)$ for 

Remarks: (a) no symmetry factor!

$\mathcal{L}_I$ is built-up from 3 diff. fields

(b) momentum direction of fermion line <u>important</u>!

(c) Dirac indices are contracted along fermion lines, e.g.

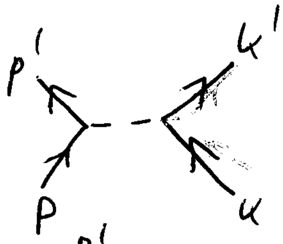


$$\Big(\ \cdots\ \Big)_{\xi\xi'} \simeq (-ih)^2 \left(\frac{i(\not p + m)}{p^2 - m^2 + i\varepsilon} \right)_{\xi\eta} (-ih) \left(\frac{i(\not p + m)}{p^2 - m^2 + i\varepsilon} \right)_{\eta\xi'}$$

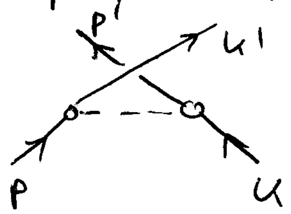
//

$$\cdots\ \bar\psi\,\phi \left[(\psi\bar\psi)\,\phi\,(\psi\bar\psi) \right]_{\xi\xi'} \phi\,\psi\ \cdots \qquad (4.100)$$

Exercise: scattering process



\+



$$\Rightarrow i\mathcal{M} = (-ih)^2 \Big[\bar u(\vec p\,') u(\vec p) \frac{i}{(p-p')^2 - m_\phi^2} \bar u(\vec k') u(\vec k)$$

$$- \bar u(\vec p\,') u(\vec k) \frac{i}{(p-k)^2 - m_\phi^2} \bar u(\vec k') u(\vec p) \Big]$$

(4.101)

QED: couple electron Ψ_e to photon A_μ:

$$\mathcal{L}_I = - e \bar{\Psi} A_\mu \gamma^\mu \psi$$

and

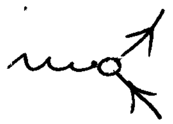
$$\mathcal{L}_{QED} = \mathcal{L}_{photon} + \mathcal{L}_{Dirac} + \mathcal{L}_I$$

with

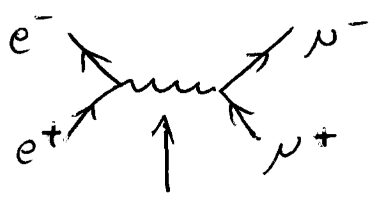
$$\mathcal{L}_{Dirac} + \mathcal{L}_I = \bar{\Psi} (i \not{D} - m) \psi$$

where

$$D_\mu = \partial_\mu - i e A_\mu$$

Vertex:  $= ie$

Example:



photon propagator $\Rightarrow$ Quantis. of gauge field