5 Gauge fields

5.1 Gauge symmetry

Consider Dirac theory of e^+, e^-

$$\mathcal{L}_D = \bar{\Psi}(x)\,(i\not{\partial} - m)\,\Psi(x) \qquad (5.1)$$

or complex scalar theory

$$\mathcal{L}_\phi = \partial_\nu \phi\, \partial_\nu \phi^* - m^2 \phi\phi^* - V(\phi\phi^*) \qquad (5.2)$$

The Lagrangians (5.1), (5.2) are invariant under global $U(1)$-rotations:

$$\begin{aligned} \Psi &\to e^{i\alpha}\Psi, & \bar{\Psi} &\to \bar{\Psi} e^{-i\alpha} \\ \phi &\to e^{i\alpha}\phi, & \phi^* &\to \phi^* e^{-i\alpha} \end{aligned} \qquad (5.3)$$

Global rotation in field space

Let us require the invariance of the theory under <u>local</u> rotations. (gauge symm.)

e.g.

$$\psi(x) \to e^{i\alpha(x)}\,\psi(x) \qquad (5.4)$$

$\mathcal{L}_D$ is not invariant,

$$\begin{aligned}\mathcal{L}_D &\to \mathcal{L}_D - \bar{\psi}(\not{\partial}\alpha)\,\psi \\ &= \mathcal{L}_D - \partial_\nu \alpha\, j^\nu\end{aligned} \qquad (5.5)$$

with $j^\nu = \bar{\psi}\gamma^\nu\psi$. Hence, if we add a term $A_\nu j^\nu$ to $\mathcal{L}_D$, and require invariance, it follows

$$\begin{aligned}\mathcal{L}_D + A_\nu j^\nu &\to \mathcal{L}_D - \partial_\nu\alpha\, j^\nu + A'_\nu j^\nu \\ &\overset{!}{=} \mathcal{L}_D + A_\nu j^\nu\end{aligned}$$

$$\Rightarrow \boxed{A_\nu \to A_\nu + \partial_\nu \alpha} \qquad (5.6)$$

Also: $\mathcal{L}$ Lorentz scalar: $A_\nu \xrightarrow{\Lambda} \Lambda^\nu{}_\nu A_\nu$ $\qquad (5.7)$

as $A_\nu j^\nu$ scalar

We write the invariant action

$$\mathcal{L}_D = \bar{\Psi}(i\not{D} - m)\Psi \qquad (5.8)$$

with covariant derivative

$$D_\mu = \partial_\mu - iA_\mu \qquad (5.9)$$

- A_μ is also called a connection, it induces covariant transformation properties for D_μ:

$$D_\mu \to e^{i\alpha(x)} D_\mu e^{-i\alpha(x)}$$

$$\Rightarrow \quad D_\mu \Psi \to e^{i\alpha(x)} D_\mu \Psi$$ transforms homog. as the field Ψ

(5.10)

as well $D_\mu \phi \to e^{i\alpha(x)} D_\mu \phi$

Similarily we get

$$\mathcal{L}_\phi = D_\mu \phi (D_\mu \phi)^* - m^2 \phi\phi^* - V(\phi\phi^*) \qquad (5.11)$$

is invariant under $\phi(x) \to e^{i\alpha(x)}\phi(x)$

Dynamics of gauge field A_μ:

(i) Construct gauge-invariant scalar quantities from A_μ.
This is easily done from D_μ, which transforms covariantly:

$$i[D_\mu, D_\nu] = (\partial_\mu A_\nu - \partial_\nu A_\mu) = F_{\mu\nu} \quad (5.12)$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

(5.13)

Lorentz trafos: A_μ transforms as vector, see eq. (5.7)

$\Rightarrow F_{\mu\nu} \to \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}$

transforms as tensor

$F_{\mu\nu}$: field strength, curvature

gauge invariant: $F_{\mu\nu} \to F_{\mu\nu}$ (5.14)

with $D_\mu \to e^{i\alpha} D_\mu e^{-i\alpha}$

In summary: $F_{\mu\nu}$ gauge inv.

$F_{\mu\nu} F^{\mu\nu}$ gauge inv. Lorentz scalar

$\Rightarrow$ Gauge-inv. Lagrangian:

$$\mathcal{L}_{QED} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_D \qquad (5.15)$$

Reparameterise $A_\mu \to e A_\mu$

$\uparrow$ electric charge

$$\Rightarrow \mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i \not{D} - m)\psi \qquad 5.16$$

with $D_\mu = \partial_\mu - ie A_\mu$

Remark: construction also goes through for $\psi \to U \psi$ for $U \in SU(N)$

$$F_{\mu\nu} F^{\mu\nu} \to \sum_{a=1}^{N^2-1} F_{\mu\nu}^a F^{\mu\nu a}$$

with $F_{\mu\nu} = i/e \, [D_\mu, D_\nu] \sim [A_\mu, A_\nu]$