## 5.2 Quantisation

We concentrate on the pure gauge field Lagrangian:

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{5.17}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

EoM:

$$\partial_\mu F^{\mu\nu} = \left(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma\right) A_\sigma = 0$$

(5.18)

with current $\partial_\mu F^{\mu\nu} = J^\nu$

Eq. (5.18) reflects the redundancy of the gauge field $A_\mu$: $A_\mu \to A_\mu + e\,\partial_\mu \alpha$

$$\left(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma\right) \partial_\sigma \alpha(x) = 0 \tag{5.19}$$

Problems:

(i) The eqs. (5.18), (5.19) already entail that $A^\nu$ cannot have canonical commutation relations! What about the canonical momentum $\pi^\nu$:

$$\pi^\nu = \frac{\partial \mathcal{L}}{\partial \partial_0 A_\nu} = -\frac{1}{4} \frac{\partial}{\partial \partial_0 A_\nu} \left( F_{\rho\sigma} F_{\gamma\delta} \eta^{\sigma\delta} \eta^{\rho\gamma} \right)$$

$$= -\frac{1}{2} F_{\rho\sigma} \eta^{\sigma\delta} \eta^{\rho\gamma} \frac{\partial F_{\gamma\delta}}{\partial \partial_0 A_\nu}$$

$$= F^{\nu 0} \qquad (5.20)$$

In particular: $\boxed{\pi^0 = 0}$ ⇐ reflects redundancy

(ii) Remove redundancy by fixing the gauge, e.g. Lorentz- or covariant gauge:

$$\partial_\nu A^\nu = 0 \qquad (5.21)$$

Remark: For $A^\nu$ with (5.21) we can write

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi}(\partial_\nu A^\nu)^2 \qquad (5.22)$$

or $$S[A] = \frac{1}{2}\int d^4x \; A_\mu \left(\partial_\rho \partial^\rho \eta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu \partial^\nu\right) A_\nu \qquad (5.23)$$

and EoM $$\partial_\mu F^{\mu\nu} = -\frac{1}{\xi}\partial^\nu (\partial_\mu A^\mu)$$

Note that $\left(\partial_\rho \partial^\rho \eta^{\mu\nu} - (1-\frac{1}{\xi})\partial^\mu\partial^\nu\right)$ is invertible, it is specifically simple for $\boxed{\xi = 1}$.

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With the gauge (5.21) (or $\xi = 1$) the EoM read

$$\boxed{\partial_\rho \partial^\rho A^\nu = 0} \qquad (5.24)$$

KG-equation

Eq. (5.24) suggest a quantised field

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \left\{ e^{-ikx} a_\mu(\vec{k}) + e^{ikx} a_\mu^\dagger(\vec{k}) \right\} \tag{5.25}$$

with commutation relations

$$\boxed{\left[ a_\mu(\vec{k}), a_\nu^\dagger(\vec{k}') \right] = -\boxed{\eta_{\mu\nu}} (2\pi)^3 \, \delta(\vec{k}-\vec{k}')} \tag{5.26}$$

necessary for Lorentz-sym.

and $\left[ a_\mu(\vec{k}), a_\nu(\vec{k}') \right] = 0 = \left[ a_\mu^\dagger(\vec{k}), a_\nu^\dagger(\vec{k}') \right]$

However eqs. (5.25), (5.26) are not compatible with eq. (5.21):

$$\partial_\mu A^\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{-i}{\sqrt{2k^0}} \left\{ e^{-ikx} k^\mu a_\mu(\vec{k}) + e^{ikx} k^\mu a_\mu^\dagger(\vec{k}) \right\}$$

$$\overset{!}{=} 0 \tag{5.27}$$

This entails that $k^\mu a_\mu(\vec{k}) \overset{!}{=} 0$. Note that if (5.27) fails, the EoM is <u>not</u> satisfied: $\boxed{\partial_\mu F^{\mu\nu} = -\partial^\nu \partial_\mu A^\mu}$

However

$$k^\mu [a_\mu(\vec{k}), a^+_\nu(\vec{k}')] = - k^\nu (2\pi)^3 \, \delta(\vec{k}-\vec{k}') \neq 0 \qquad (5.28)$$

Indeed one can show that it is <u>not</u> possible to quantise the gauge field $A_\mu$ with cano. com. relations and $\partial_\mu A^\mu = 0$, or other gauge conditions: If using $A^\mu$ in (5.25), (5.26), the gauge $\partial_\mu A^\mu$ has to be implemented on the states!

(iii) Fockspace $\mathcal{F}$: standard construction based on (5.25), (5.26)

(a) vacuum $|0\rangle$ with $\langle 0|0\rangle = 1$

(b) one-particle states: $\sqrt{2k^0}\, a^\dagger_\mu(\vec{k})|0\rangle$

with norm $\langle 0| a_\nu(\vec{k}')\, a^\dagger_\mu(\vec{k})|0\rangle \sqrt{2k^0 2k'^0}$

$$= -\boxed{\eta_{\mu\nu}}(2\pi)^3\, 2k^0\, \delta^3(\vec{k}-\vec{k}') \tag{5.29}$$

$\mu = \nu = i$ : positive norm states

$\mu = \nu = 0$ : negative norm states

$\Rightarrow \tilde{\mathcal{F}}$ is not the phys. Hilbert space $\mathcal{H}$, as it does not allow for prob. interpertation.

Remarks:

(i) $\eta_{\mu\nu} \to -\eta_{\mu\nu}$ does not solve the problem of negative norm states (leave aside the wrong commutators $[A^i, \pi^i]$.

(ii) Separating the positive norm sub space of $\tilde{\mathcal{F}}$ will resolve all problems (i)-(iii)

Gupta - Bleuler quantisation:

(i) We demand EoM satisfied on phys. states,

$$\boxed{\langle \text{phys. states}' |\, \partial_\nu F^{\nu\nu} \,| \text{phys. states} \rangle \overset{!}{=} 0} \quad (5.30)$$

that is, its matrix elements vanish.

Eq. (5.30) is satisfied for

$$\boxed{k^\nu a_\nu(\vec{k})\,|\text{phys. states}\rangle = 0} \quad (5.31)$$

(trivially satisfied on the vacuum)

The above suggests to rewrite $A_\nu$ in (5.25)

as

$$\boxed{\begin{aligned} A_\nu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \sum_{\lambda=0}^{3} \Big\{ & \alpha_\lambda(\vec{k})\, \varepsilon^\lambda{}_\nu(k)\, e^{-ikx} \\ & + \alpha_\lambda^\dagger(\vec{k})\, \varepsilon^{\lambda *}{}_\nu(k)\, e^{ikx} \Big\} \end{aligned}} \quad (5.32)$$

where the $\varepsilon^\lambda{}_\nu$ introduce unitary rotations from $a_\nu$ to $\alpha_\lambda$ with

$$\varepsilon^\lambda{}_\nu(k)\,\varepsilon^{\lambda'\nu *}(k) = \eta^{\lambda\lambda'}$$
$$\varepsilon^\lambda{}_\nu(k)\,\varepsilon_{\lambda\nu}{}^{*}(k) = \eta_{\nu\nu} \qquad (5.33)$$

and hence

$$\boxed{\alpha_\lambda(\vec{k}) = a_\nu(\vec{k})\,\varepsilon_\lambda{}^\nu(k) \qquad (5.34)}$$

We choose $k\cdot\varepsilon^0 = k^0 = k\cdot\varepsilon^3$ and $k\cdot\varepsilon^i = 0,\ i=1,2$

The $\varepsilon$'s are called polarisation vectors.

Eq. (5.31) now reads with $\alpha_\pm = \frac{1}{\sqrt{2}}(\alpha_0 \pm \alpha_3)$

$$\boxed{\alpha_+\,|\text{phys. states}\rangle = 0} \qquad (5.35)$$

with $\alpha_0 + \alpha_3 \simeq k^\nu a_\nu$.

In the frame with $(k^\nu) = (k^0, 0, 0, k^0)$ we have

$$(\varepsilon^\lambda)_\nu = \delta^\lambda{}_\nu \qquad (5.36)$$

The $\alpha$'s have the same commutation relation as the $a$ (unitary rotation, see also (5.33), (5.34))

It follows with

$$i=1,2: \quad [\alpha_i(\vec{k}), \alpha_i^\dagger(\vec{k}')] = (2\pi)^3 \delta(\vec{k}-\vec{k}')$$

$$[\alpha_+(\vec{k}), \alpha_-^\dagger(\vec{k}')] = -(2\pi)^3 \delta(\vec{k}-\vec{k}')$$

$$[\alpha_\pm(\vec{k}), \alpha_\pm^{(\dagger)}(\vec{k}')] = 0 = [\alpha_\pm(\vec{k}), \alpha_i^{(\dagger)}(\vec{k}')] \tag{5.37}$$

Physical Hilbert space $\mathcal{H}$:

(i) Physical sub-space $\mathcal{F}_{phys}$: $|\Psi\rangle \in \mathcal{F}_{phys}$

$$\Rightarrow \boxed{\alpha_+ |\Psi\rangle = 0}$$

It follows

$$|\Psi\rangle \in \mathcal{F}_{phys} \Rightarrow \alpha_i^\dagger |\Psi\rangle \in \mathcal{F}_{phys}, \; i=1,2$$

$$\text{with } \alpha_+ \alpha_i^\dagger |\Psi\rangle = \alpha_i^\dagger \alpha_+ |\Psi\rangle = 0 \tag{5.38}$$

Also $\quad \alpha_+^\dagger |\Psi\rangle \in \mathcal{F}_{phys}$

$$\text{with } \alpha_+ \alpha_+^\dagger |\Psi\rangle = \underbrace{\alpha_+^\dagger \alpha_+ |\Psi\rangle}_{0} \tag{5.39}$$

Finally $\alpha_-^+|\Psi\rangle \notin \tilde{\mathcal{F}}_{phys}$

since $\alpha_+ \alpha_-^+|\Psi\rangle = \underbrace{\alpha_-^+ \alpha_+|\Psi\rangle}_{0} + \underbrace{[\alpha_+, \alpha_-^+]|\Psi\rangle}_{\sim|\Psi\rangle \neq 0}$

(5.40)

We conclude that

$$\tilde{\mathcal{F}}_{phys} = \text{Span}\left(a_+^{+\,n_+} a_1^{+\,n_1} a_2^{+\,n_2}|0\rangle\right) \quad (5.41)$$

(ii) $\tilde{\mathcal{F}}_{phys}$ contains only states with semi-positive norm: $\langle\Psi|\Psi\rangle \geq 0$

Indeed

$$\|a_+^+|\Psi\rangle\|^2 = \langle\Psi|a_+ a_+^+|\Psi\rangle$$
$$= \langle\Psi|a_+^+ a_+|\Psi\rangle = 0 \quad (5.42)$$

and

$$\|a_1^{+\,n_1} a_2^{+\,n_2}|0\rangle\| > 0$$

with $[a^+_i, a_i] = +(2\pi)^3 2k^0 \delta$ (5.43)

(iii) We identify two states $|\psi_1\rangle, |\psi_2\rangle$ with $\| |\psi_1\rangle - |\psi_2\rangle \| = 0$: every matrix element of an operator $\mathcal{O}(a_i^{(+)}, a_+^{(+)})$ vanishes $\langle \psi | \mathcal{O}(|\psi_1\rangle - |\psi_2\rangle)$ vanishes.

$\Rightarrow$ We define the physical Hilbert space as the space of equivalence classes

$$\mathcal{H} = \mathcal{F}_{phys} / \sim \qquad (5.44)$$

with $|\psi_1\rangle \sim |\psi_2\rangle$ for $\| |\psi_1\rangle - |\psi_2\rangle \| = 0$

We have for $|\psi\rangle \in \mathcal{H}$

$$\langle \psi | \psi \rangle > 0 \quad \text{for } |\psi\rangle \neq 0$$

$$(5.45)$$

$$\alpha_+ |\psi\rangle = 0$$

and hence $\langle \psi' | \partial_\mu F^{\mu\nu} | \psi \rangle = 0$

with $\langle \psi' | \partial_\mu F^{\mu\nu} | \psi \rangle = \langle \psi' | \partial^\nu \partial_\mu A^\mu | \psi \rangle = 0 \qquad (5.46)$

Feynman rules:

(i) Propagator: $x^0 > y^0$

$$\langle 0 | \; A_\mu(x) \, A_\nu(y) | 0 \rangle$$

$$= \langle 0 | \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2k'^0} \, e^{-ikx + ik'y} \left[ a_\mu(\vec{k}), a^\dagger_\nu(\vec{k}') \right] | 0 \rangle$$

$$= -\eta_{\mu\nu} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} \, e^{-ik(x-y)} \qquad (5.47)$$

$$\Rightarrow \langle 0 | T A_\mu(x) \, A_\nu(y) | 0 \rangle = -\eta_{\mu\nu} \underbrace{D_F(x-y)}_{\text{scalar prop}} \qquad (5.48)$$

or $$\underset{\mu \quad k \quad \nu}{\sim\!\sim\!\sim} = - \frac{i \eta_{\mu\nu}}{k^2 + i\varepsilon}$$

(ii) final states/initial states

$$|\vec{k}, \varepsilon\rangle = \alpha^{+}(\vec{k})|0\rangle \qquad (5.49)$$

Note that $\alpha^{+} = \varepsilon^{*}_{\nu}\, a^{+\nu}$, eq. (5.34).

Hence we have

$$A_{\nu}(x)\Big|_{\text{annih.}} |\vec{k}, \varepsilon\rangle = \int \frac{d^3k'}{(2\pi)^3} \sqrt{\frac{2k^{0'}}{2k^0}}\, e^{-ik'x}\, a_{\nu}(\vec{k}')\, \alpha^{+}(\vec{k})|0\rangle$$

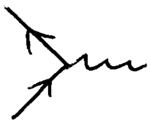
$$= \varepsilon^{*}_{\nu}(k) \qquad (5.50)$$

that is $\boxed{A|\vec{k}, \varepsilon\rangle := \varepsilon^{*}}$

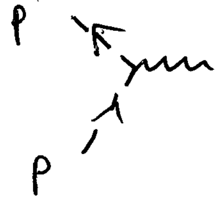
$$(5.51)$$

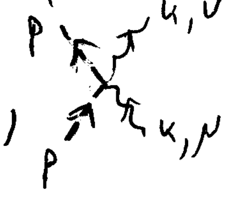
and $\langle \vec{k}, \varepsilon| A = \varepsilon$

(iii) Vertices:

(a) $\mathcal{L}_I = e\bar{\psi}\not{A}\psi$ :  $= ie\gamma^{\nu}$ (5.52)

(b) $\mathcal{L}_I = D_{\nu}\phi (D^{\nu}\phi)^{*} - \partial_{\nu}\phi\, \partial^{\nu}\phi^{*}$ (5.53)



$= -ie\,(p_{\nu} + p'_{\nu})$ ,


$= 2ie^2 \eta^{\mu\nu}$

# Gauge independence & Feynman rules

We can add a longitudinal peace to the field $A_\mu$ without changing physics!

$$A_\mu \to A_\mu + \alpha\, \partial_\mu \frac{1}{\partial_\rho \partial^\rho} \partial^\nu A_\nu \qquad (5.54)$$

or in Lagrangian

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2 \qquad (5.55)$$

$\Rightarrow$ Propagator:

$$\langle 0 | T A_\mu A_\nu | 0 \rangle (p^2)$$

$=$ 

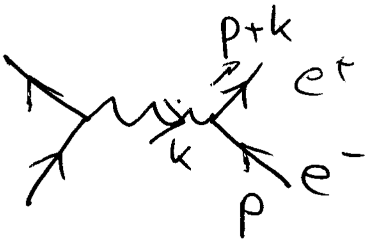
$$= -i \left( \frac{\eta_{\mu\nu}}{k^2 + i\varepsilon} - (1-\xi) \frac{k_\mu k_\nu}{(k^2 + i\varepsilon)^2} \right)$$

$$= -\frac{i}{k^2 + i\varepsilon} \left( \eta_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2 + i\varepsilon} \right)$$

(5.56)

$\xi$ drops out of the scattering amplitudes

E.g.:




$$\simeq \langle 0|T A_\mu A_\nu|0\rangle(k)\, \bar v(p+k)\gamma^\nu u(p) \qquad (5.57)$$

We use:

$$\xi k_\nu k_\mu \, \bar v(p+k)\gamma^\mu u(p)$$

$$= \xi k_\nu \bar v(p+k) \not k \, u(p)$$

$$= \xi k_\nu \bar v(p+k)(\not k + \not p - \not p)\, u(p)$$

p. 116: $\underbrace{(\not p - m)u(p)}_{=0} \Rightarrow$ $= \xi k_\nu \bar v(p+k)(\not k + \not p - m)\, u(p)$

$\downarrow$

$\underbrace{\bar v(p+k)(\not p + \not k - m)}_{=0}$ $= 0$ $\qquad \square$

(5.58)

Gauge invariant observables:

e.g. E and B - fields

$$E^i = - F^{0i} = -(\partial^0 A^i - \partial^i A^0)$$

and

$$B^i = \varepsilon^{ijk} F_{jk} \qquad (5.59)$$

They read ($A_\nu$ on page 143, (5.32))

$$\vec{E} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}}\, ik^0 \Big\{ \Big(\vec{a} - \frac{\vec{k}}{k^0} a^0\Big) e^{-ikx} - \Big(\vec{a}^{\,+} - \frac{\vec{k}}{k^0} a^{0+}\Big) e^{ikx} \Big\}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}}\, ik^0 \Big\{ (\varepsilon_1 \cdot \alpha_1 + \varepsilon_2 \alpha_2)\, e^{-ikx} - (\varepsilon_1 \alpha_1^+ + \varepsilon_2 \alpha_2^+)\, e^{ikx} \Big\}$$

phys. pol.

$$- \underbrace{\int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}}\, ik^0 \Big\{ \frac{\vec{k}}{k^0} \alpha_+ e^{-ikx} - \frac{\vec{k}}{k^0} \alpha_+^+ e^{ikx} \Big\}}_{\sim 0} \qquad (5.60)$$

Analogously

$$B^i(x) = \varepsilon^{ijl} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} ik^j \left\{ \varepsilon^l \cdot \alpha \, e^{-ikx} - \varepsilon^l \alpha^+ e^{ikx} \right\} \tag{5.61}$$

It follows that only $\alpha_{1/2}$ and $\alpha_+$ appear in $\vec{E}$ and $\vec{B}$. Sandwiched between physical states $|\psi\rangle \in \mathcal{H}$, $\alpha_+$ drops out.

The Hamiltonian reads, with $\pi^i = E^i$,

$$H = \int d^3x \left\{ \vec{\pi} \cdot \partial_0 \vec{A} + \underbrace{\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{1/2(\vec{E}^2 - \vec{B}^2)} \right\}$$

$$= \int d^3x \left\{ \frac{1}{2} (\vec{E}^2 + \vec{B}^2) + \vec{\nabla}(\vec{E} A_0) \right\}$$

$\vec{\nabla}\vec{E} = 0$

$$= \frac{1}{2} \int d^3x \left\{ \vec{E}^2 + \vec{B}^2 \right\} \tag{5.62}$$

where we have used

$$\vec{\nabla} \vec{E} = \left( -\partial^0 \partial^i A^i + \partial^{i2} A^0 \right)$$

$$\partial_\mu A^\mu = 0 \longrightarrow \quad = \left( -\partial^{02} + \partial^{i2} \right) A^0 = 0$$

We insert the E-B-field operators eqs. (5.60), (5.61) arrive at

$$p^0 = H \simeq \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\frac{k_0}{2k^0}\,k^0 \sum_{i=1}^{2}\left(\alpha_i\,\alpha_i^+ + \alpha_i^+\alpha_i\right)$$

$$\simeq \int \frac{d^3k}{(2\pi)^3}\,k^0 \sum_{i=1}^{2} \alpha_i^+(\vec{k})\;\alpha_i(\vec{k}) \qquad (5.63)$$

where we have dropped the $\alpha_+$-terms, and, in the second line, the vacuum terms.

Similarily we get for $\vec{P}$:

$$\vec{P} = \int d^3x\;\vec{E}\times\vec{B}$$

$$\simeq \int \frac{d^3k}{(2\pi)^3}\,\vec{k}\,\sum_{i=1}^{2}\alpha_i^+(\vec{k})\,\alpha_i(\vec{k}) \qquad (5.64)$$

where we have dropped the vacuum terms.