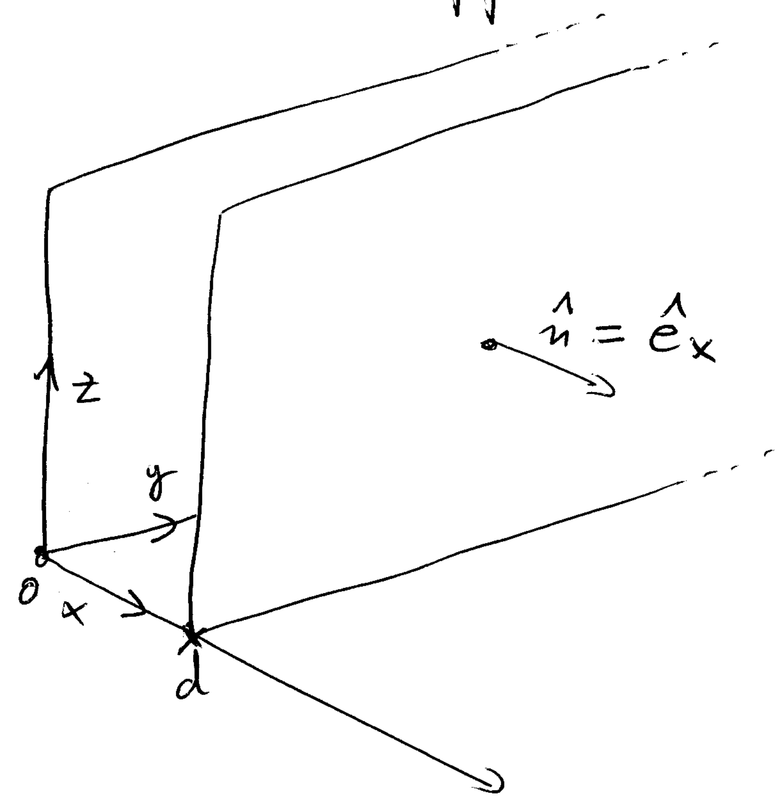
Casimir effect:



Exp.: Lamoreaux et al '97
PRL 78 (1997) 5

Mohideen, Roy '98
PRL 81 (1998) 4549

Solution for $\vec{E}, \vec{B}$: plane waves

$$\vec{E} \simeq \vec{\varepsilon}\, e^{\pm ikx}, \quad \vec{B} \simeq \vec{k} \times \vec{E} \qquad (5.65)$$

polarisation

with boundary conditions:

$$\hat{n} \times \vec{E}\Big|_{x=0,d} = 0$$

$$\hat{n} \times \vec{B}\Big|_{x=0,d} = 0 \qquad (5.66)$$

The electric (magnetic) fields parallel to the plates vanish on the plates!

Solution to eq (5.66):

$$\vec{E} \simeq \vec{\varepsilon}\, \sin k_x x \cdot e^{i(k_y y + k_z z - k^0 t)} \qquad (5.67)$$

with $k^0 = \sqrt{\vec{k}^2}$

$$k_x = n\pi/d \, , \quad n = 1, 2, \ldots \qquad (5.68)$$

We compute the ground state energy: see eq. (5.63)

$$\langle 0|H|0\rangle_d = \frac{1}{2}\frac{1}{d}\sum_{n=1}^{\infty}\sum_{i=1}^{2}\int \frac{d^2 k_{\parallel}}{(2\pi)^2} k^0 \langle 0|\alpha_i(\vec{k})\alpha_i^\dagger(\vec{k})|0\rangle \qquad (5.69)$$

vacuum state dep. on distance d

with $\int_{\mathbb{R}} \frac{dk_x}{(2\pi)} \to \frac{1}{d}\sum_{n=1}^{\infty}$ and $e^{ik_x x} \to \sin k_x x$

with k_x in (5.68)

and $\vec{k}_{\parallel} = (k_y, k_z)$.

The com. rel. eq. (5.26) of the α_i's now reads

$$[\alpha_i(\vec{k}), \alpha_j^\dagger(\vec{k}')] = (2\pi)^2\, \delta^2(\vec{k}_{\parallel} - \vec{k}_{\parallel}') \cdot \underbrace{2\pi\, \delta(k_x - k_x')}_{\text{strictly speaking } d\cdot\delta_{nn'}}\, \delta_{ij} \qquad (5.70)$$

We conclude

$$\langle 0|H|0\rangle_d = \frac{1}{d}\sum_{n=1}^{\infty}\int\frac{d^2k_{\parallel}}{(2\pi)^2}\sqrt{\vec{k}_{\parallel}^2+\left(\frac{n\pi}{d}\right)^2}\,\underbrace{\left[(2\pi)^2\delta^2(0)\right]d}_{A\cdot d}$$

with $(2\pi)^2\delta^2(0) = \int_A d^2x\, e^{-i(k_{\parallel}-k_{\parallel})x}$ (5.71)

where A is the (infinite) area of the plate.

$$\Rightarrow \langle 0|H|0\rangle_d = A\sum_{n=1}^{\infty}\int\frac{d^2k_{\parallel}}{(2\pi)^2}\sqrt{\vec{k}_{\parallel}^2+\left(\frac{n\pi}{d}\right)^2} \qquad (5.72)$$

We do not have to worry about the area factor, which is well-defined for finite plates (which then makes the formula an approximate one).

The other infinity comes from large momenta (UV). There, however, it is the by now common infinite vacuum energy which we may subtract.

In order to have well-defined quantities, we cut-off or regularise large momenta or high energies in eq. (5.72).

$$\langle 0|H|0\rangle_d \longrightarrow \hbar \sum_n \int \frac{d^2k_{\parallel}}{(2\pi)^2} \sqrt{\vec{k}_{\parallel}^2 + \left(\frac{n\pi}{d}\right)^2} \cdot r_\Lambda\left(\vec{k}_{\parallel}^2 + \left(\frac{n\pi}{d}\right)^2\right) \tag{5.73}$$

with $r_\Lambda(x \gg \Lambda^2) \to 0$, $r_\Lambda(x \ll \Lambda^2) \to 1$.

and hence

$$\boxed{\begin{aligned} E_{d,r_\Lambda} &= \langle 0|H|0\rangle_{d,r_\Lambda} \\ &= \hbar/2\pi \sum_{n=1}^{\infty} R_\Lambda(n) \end{aligned}} \tag{5.74}$$

with $R_\Lambda(n) = \int_0^\infty dk_{\parallel}\, k_{\parallel} \sqrt{k_{\parallel}^2 + \left(\frac{n\pi}{d}\right)^2}\; r_\Lambda\left(k_{\parallel}^2 + \left(\frac{n\pi}{d}\right)^2\right)$

This energy cannot be measured; what can be measured, are energy differences. To that end we take the infinite distance limit, $d \to \infty$.

Then, $E_{d\to\infty, r_1}$ tends towards the standard (regularized) vacuum energy, which we have set to zero. The energy difference of the situation without plates, $d=\infty$, and that with plates, is

$$\boxed{\begin{aligned}\Delta E_{d,r_1} &= E_{d,r_1} - E_{\infty,r_1}\cdot\frac{V_d}{V_\infty}\\ &= \frac{\hbar}{2\pi}\left[\sum_{n=1}^{\infty} R_1(n) - \int_0^\infty dn\, R_1(n) + \frac{1}{2}R_1(0)\right]\end{aligned}} \quad (5.75)$$

The integral in (5.75) can be turned into a sum (Euler-MacLaurin formula):

$$\int_0^\infty dn\, R_1(n) = \sum_{n=1}^{\infty}\left\{R_1(n) + \frac{1}{(2n)!}B_{2n}R_1^{(2n-1)}(0)\right\} + \frac{1}{2}R_1(0) \qquad (5.76)$$

with Bernoulli numbers B_{2n}.

We finally get

$$\boxed{\Delta E_{d,r_1} = \frac{\hbar}{2\pi} \sum_{n=1}^{\infty} \frac{1}{(2n)!} B_{2n} R_1^{(2n-1)}(0)} \qquad (5.77)$$

and with $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \ldots$

$$\Delta E_{d,r_1} = \frac{\hbar}{2\pi} \left\{ -\frac{1}{12} R_1^{(1)}(0) + \frac{1}{720} R_1^{(3)}(0) + \ldots \right\}$$

(5.78)

We use $k = \sqrt{k_{\parallel}^2 + \left(\frac{\pi n}{d}\right)^2} \Rightarrow dk\, k = dk_{\parallel}\, k_{\parallel}$

$$R_1(n) = \int_{n\pi/d}^{\infty} dk\, k^2\, r_1(k^2) \qquad (5.79)$$

Thence:

$$R_1^{(1)}(n) = -\pi/d \left(\frac{n\pi}{d}\right)^2 r_1\left((n\pi/d)^2\right)$$

$$R_1^{(2)}(0) = 0$$

$$R_1^{(3)}(0) = -2\left(\pi/d\right)^3 \qquad (5.80)$$

and $R_1^{(i>3)}(0)$ depend on r_1 and $R_1^{(i>3)} \sim \left(\frac{1}{\Lambda d}\right)^{i-3}$

It follows for $\Lambda \to \infty$: $\Delta E_d = \Delta E_{d,1}$

$$\boxed{\Delta E_d = -\frac{\pi^2}{720}\, \hbar / d^3} \qquad (5.81)$$

Remarks:

(i) The energy difference ΔE_d is finite and independent of the regularisation procedure!

(ii) The Force/area is computed from the energy-density $\Delta \mathcal{E}_d = \Delta E_d / A = -\frac{\pi^2}{720} \, 1/d^3$

$$\Rightarrow \text{Force/area} \quad f = -\frac{\partial \Delta \mathcal{E}_d}{\partial d}$$

$$= -\frac{\pi^2}{240} \frac{1}{d^4} (\hbar c)$$

$$\Rightarrow \quad \boxed{f \simeq -1.3 \cdot 10^{-27} \, \text{Pa} \, \frac{m^4}{d^4}}$$

(5.82)