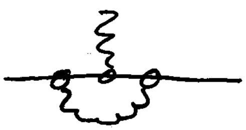
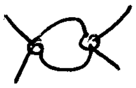
7 Renormalisation

So far, we have only considered tree-level diagrams. We have, however, seen in the last chapter, that physically interesting effects, are related to loop diagrams, for example the Landé factor, p. 169:  or non-linear QED, .

In chapter 3.5 (LSZ) we have also discussed Z-factors in propagators and vertices (at the example of the ϕ^4-theory, with diagrams , , ...

Last but not least we have 'normalised' the Hamiltonian by normal-ordering, ~ .

7.1 ϕ^4 - theory

Action of ϕ^4 - theory:

$$S[\phi] = -\frac{1}{2}\int d^4x\ \phi_0 \left(\partial^2 + m_0^2\right)\phi_0 - \frac{\lambda_0}{4!}\int d^4x\ \phi_0^4 \qquad (7.1)$$

with <u>bare</u> fields ϕ_0 and parameters/couplings m_0^2, λ_0. We write

$$\phi_0 = Z_\phi^{1/2}\,\phi$$
$$m_0^2 = Z_m\, m^2$$
$$\lambda_0 = Z_\lambda\, \lambda \qquad (7.2)$$

with <u>renormalised</u> or physical fields ϕ and parameters m^2, λ, and <u>multiplicative</u> renormalisations Z_ϕ, Z_m, Z_λ.

The Z's are expanded in powers of λ:

$$Z = \underset{\text{classical}}{1} + \underset{\text{quantum}}{\delta Z}\ , \quad \delta Z = \delta Z_1 \lambda + \delta Z_2 \lambda^2 + \ldots \qquad (7.3)$$

Remember LSZ, eq. (3.103), p. 87 with fields ϕ_0

$$\langle T\, \phi_0 \phi_0 \rangle(p^2)\Big|_{pole} = \frac{iZ}{p^2 - m^2_{phys}} + \text{finite}$$

$$= Z_\phi \langle T\, \phi\, \phi \rangle\Big|_{pole} \qquad (7.4)$$

We demand $Z_\phi = Z$ and hence

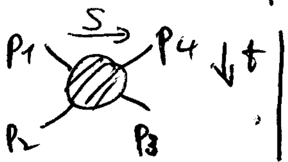
$$\langle T\, \phi\, \phi \rangle(p^2)\Big|_{pole} = \frac{i}{p^2 - m^2} + \text{finite} \qquad (7.5)$$

where we have implicitly fixed Z_ϕ such that $\boxed{m^2 = m^2_{phys}}$. Eq. (7.4) and (7.5) can be cast into the form

$$\boxed{\begin{aligned} &\left[\langle T\, \phi\, \phi \rangle(p^2)\right]^{-1}_{p^2=m^2} = 0 && (7.6a) \\ &i\,\partial_{p^2}\left[T\, \phi\, \phi \rangle(p^2)\right]^{-1}_{p^2=m^2} = 1 && (7.6b) \end{aligned}}$$

This fixes the constants Z_ϕ and Z_m. More generally we fix $\langle T\, \phi\phi \rangle$ at some scale $p^2 = \mu^2$; μ is called renormalisation scale.

The coupling renormalisation Z_λ is fixed by fixing the amputated four-point function:



$$\Big|_{s^2=t=u=m^2} = -i\lambda \leftarrow \text{symmetric point}$$

in terms of Green fct: (using eq. (7.5))

$$\boxed{\prod_i \left[\langle T\phi\,\phi\rangle(p_i)\right]^{-1} \cdot \langle T\phi(p_1)\cdots\phi(p_4)\rangle\Big|_{s^2=u=t=m^2} = -i\lambda \quad \leftarrow \lambda=\lambda_{\text{phys}}\big|_{\text{sym. point}} \qquad (7.6c)}$$

The Eqs. (7.6) are called renormalisation conditions. They fix the map between the bare quantities ϕ_0, m_0, λ_0 to the renormalised (finite) quantities ϕ, m, λ.

Remark: (i) The finiteness of correlation functions of the renormalised fields Φ follows from the finiteness of (7.5a–c). Hence the Z's have to cancel the loop divergencies.

(ii) In (perturbatively) renormalisable theories it is sufficient to introduce the Z's (and similar quantities) for getting a manifestly finite theory

(iii) The freedom of (re)-normalising fields and couplings also encodes that Green functions are <u>not</u> by themselves physical observables.

For example, we could have renormalised

the theory at some other momentum scale, $p^2 = \mu^2$ with the conditions (7.6), with

$$\lambda = \lambda_{phys}\big|_{p^2=\mu^2} \qquad \qquad (7.6d)$$

$$m^2 = m^2_{phys}\big|_{p^2=\mu^2}$$

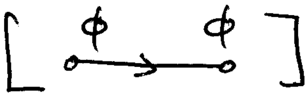
Physics is invariant under changing μ, hence

$$\boxed{\mu \frac{d}{d\mu}(\text{Phys. Observables}) = 0} \qquad (7.7)$$

Eqs. (7.6) encodes the reparameterisation invariance of the theory & the insensitivity of physics to the specific <u>renormalisation scheme</u>. μ is called <u>renormalisation group (RG) scale</u>. The generator of the RG is $\mu \frac{d}{d\mu}$, the RG is a one-parameter, Abelian semi group. (See QFT II)

Feynman rules in terms of renormalised quantities:

Prop.:

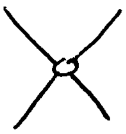
$$\Big[$$  $$\Big]^{-1} = Z_\phi \; \frac{p^2 - Z_m m^2}{i}$$

$$= \left[\frac{i}{p^2 - m^2}\right]^{-1} + \underbrace{(-i)\left[(1 - Z_\phi)\, p^2 - (1 - Z_\phi Z_m)\, m^2\right]}$$

(7.8)

$$= -i\left[\underset{<1}{(1 - \delta_\phi)}\; p^2 - (1 - Z_\phi Z_m)\, m^2\right] \tag{7.9}$$

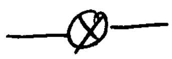
Vertex:



$$= -i\, Z_\lambda Z_\phi^2 \lambda = -i\lambda + \underbrace{i(1 - Z_\phi^2 Z_\lambda)\lambda}$$

(7.10)

$$= i\lambda\,(1 - Z_\phi^2 Z_\lambda) \tag{7.11}$$

 ,  : Counter terms

Renormalisation at one loop:

(1) Mass correction: (see p. 94)

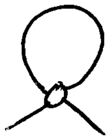
 $=$  $+\frac{1}{2}$  $+\ \mathcal{O}(\lambda^2)$

$$= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left[-i\,\Pi(p)\right]\frac{i}{p^2-m^2} + \dots \qquad (7.12)$$

with $-i\,\Pi(p) = \Big[\frac{1}{2}$  $+$  $\Big]$

$= \Big[\frac{1}{2}$  $\underbrace{- i(1-Z_\phi)\,p^2 + i(1-Z_\phi Z_m)\,m^2\Big]}_{\text{finite}}$ (7.13)

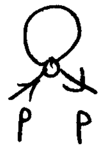
Diagram:

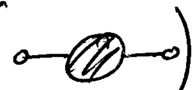
 $= -i\lambda \int \frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2}$ (7.14)

$$\Rightarrow\ -i\,\Pi(p) = -\frac{i\lambda}{2}\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2} + i(1-Z_\phi Z_m)\,m^2$$

$$- i(1-Z_\phi)\,p^2 \qquad (7.15)$$

Remarks

(i)  has no dependence on external momentum p $\Rightarrow$ $Z_\phi\big|_{1\text{-loop}} = 1$

(ii) $\Big($  $\Big)^{-1}\Big|_{p^2=m^2} = 0$

renorm. cond. eq. (7.6a) $\Rightarrow$ $\Pi(p)\big|_{1\text{-loop}} \stackrel{!}{=} 0$ (7.16)

We conclude that

$$\boxed{1 - Z_m = \frac{1}{2}\frac{\lambda}{m^2}\int \frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2}} \qquad (7.17)$$

It is left to compute

$$\int \frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2} \qquad (7.18)$$

Two problems: (a) Integrand diverges on-shell $q^2 = m^2$

(b) Integral diverges for $q^2 \to \infty$

Resolution:

ⓐ Wick rotation: $q_\mu^0 = i\, q_E^0$ (7.19)

see p. 186a

$$\Rightarrow \quad q_{\mu\,\nu}\, q_\mu^{\ \nu} = -\, q_{E\,\nu}\, q_{E\,\nu}$$

(7.20)

$$\eta_E^{\nu\nu} = -\,\mathbb{1}$$

$$\Rightarrow \int \frac{d^4 q_\mu}{(2\pi)^4}\, \frac{i}{q_\mu^2 - m^2} = \int \frac{d^4 q_E}{(2\pi)^2}\, \frac{1}{q_E^2 + m^2} \qquad (7.21)$$

ⓑ Regularisation: 2 Examples

(1) Momentum cut-off Λ:

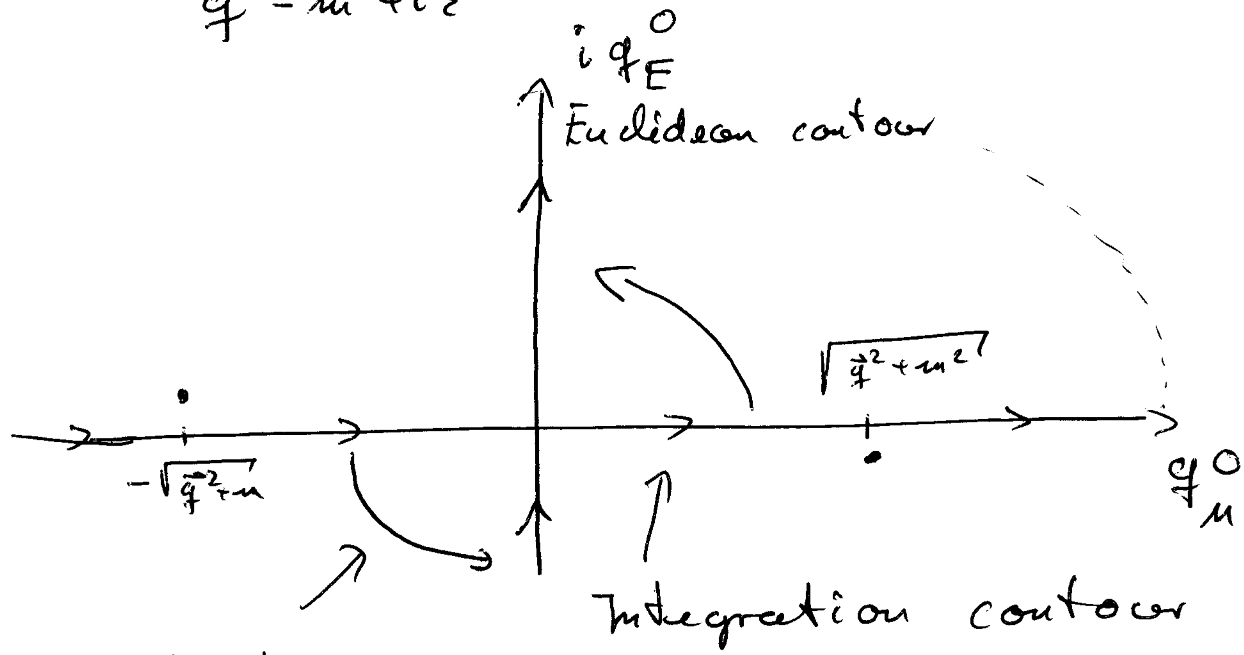
$$\int_{\mathbb{R}^4} \frac{d^4 q}{(2\pi)^4} \rightarrow \int_{q^2 \le \Lambda^2} \frac{d^4 q}{(2\pi)^4} \qquad (7.22)$$

Then $$\int_{q^2\le\Lambda^2} \frac{d^4 q}{(2\pi)^4}\, \frac{1}{q^2+m^2} = \frac{1}{8\pi^2} \int_0^\Lambda dq\, \frac{q^3}{q^2+m^2} \qquad (7.23)$$

$$= \frac{1}{16\pi^2}\left[\Lambda^2 + m^2 \ln \frac{m^2}{\Lambda^2+m^2}\right]$$

Wick-rotation:
recall the $i\varepsilon$ of time-ordering:

$$\frac{1}{q^2-m^2+i\varepsilon}$$



rotate by
avoiding the poles

The rotated Euclidean contour runs iq_E^0 from $-i\infty$ to $i\infty$, or q_E^0 from $-\infty$ to ∞. Hence we set

$$q_M^0 = i q_E^0$$

$$\Rightarrow \quad q_{M\nu}\, q_M^{\nu} = - q_{E\nu}\, q_{E\nu}$$

$$\int_{\mathbb{R}^4} d^4 q_M \;\to\; i \int_{\mathbb{R}^4} d^4 q_E$$

(2) Dimensional Regularisation:

$$\int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2+m^2} \longrightarrow \underbrace{(\bar{\mu}^2)^{\frac{4-d}{2}} \int \frac{d^d q}{(2\pi)^d}}_{\text{dim } 4} \frac{1}{q^2+m^2}$$

$$= \frac{\Omega_d}{(2\pi)^d} \cdot (\bar{\mu}^2)^{\frac{4-d}{2}} \underbrace{\int_0^\infty dq\, q^{d-1} \frac{1}{q^2+m^2}}_{\text{defined for } d<2} \qquad (7.24)$$

For $d<2$ we can compute (7.24), and then we analytically extend the result. With p. 187a

$$(\bar{\mu}^2)^{\frac{4-d}{2}} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2+m^2} = \frac{1}{(4\pi)^{d/2}} \cdot \Gamma(1-d/2) \left(\frac{\bar{\mu}^2}{m^2}\right)^{2-\frac{d}{2}} \qquad (7.25)$$

We use $d = 4-2\varepsilon$ with $\varepsilon \to 0$:

$$(\bar{\mu}^2)^{\varepsilon} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2+m^2} = \frac{m^2}{16\pi^2}\left[-\frac{1}{\varepsilon} + \gamma - 1 + \ln 4\pi - \ln m^2/\bar{\mu}^2\right]$$

with $\Gamma(-1+\varepsilon) = \frac{1}{-1+\varepsilon}\Gamma[\varepsilon] = -\frac{1}{\varepsilon} + \gamma - 1 + O(\varepsilon)$

Euler-Mascheroni: $\gamma = 0.577\ldots$

$x\Gamma(x) = \Gamma(x+1)$

$$\int \frac{d\Omega_d}{(2\pi)^d} : \quad \sqrt{\pi}^{\,d} = \left(\int dx\, e^{-x^2}\right)^d$$

$$= \int d^d x\, e^{-\vec{x}^2}, \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$$

$$\int d^d x\, e^{-\vec{x}^2} = \int d\Omega_d \int_0^\infty dx\, x^{d-1}\, e^{-x^2}$$

$$= \int d\Omega_d \;\Gamma(d/2)$$

$$\Rightarrow \boxed{\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$$

We also have

$$\int_0^\infty dq\, q^{d-1} \frac{1}{(q^2+m^2)^n} = \frac{1}{2} \frac{\Gamma(d/2)\,\Gamma(n-d/2)}{\Gamma(n)} \left(\frac{1}{m^2}\right)^{n-d/2}$$

$$\Rightarrow \boxed{\int_0^\infty \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2+m^2)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \left(\frac{1}{m^2}\right)^{n-d/2}}$$

This allows us to determine $Z_m|_{1\text{-loop}}$:

(1) Cut-off regularisation: eq. (7.23)

$$\boxed{Z_m = 1 - \frac{1}{2}\underbrace{\frac{1}{16\pi^2}\lambda}_{\text{expansion coefficient}}\left(\Lambda^2/m^2 + \ln\frac{1}{1+\Lambda^2/m^2}\right)} \quad (7.28)$$

(2) Dimensional regularisation: eq (7.27)

$$\boxed{Z_m = 1 - \frac{1}{2}\frac{1}{16\pi^2}\lambda\left(-1/\varepsilon + \gamma - 1 + \ln 4\pi - \ln\frac{m^2}{\mu^2}\right)} \quad (7.29)$$

In both cases we have (at one loop)

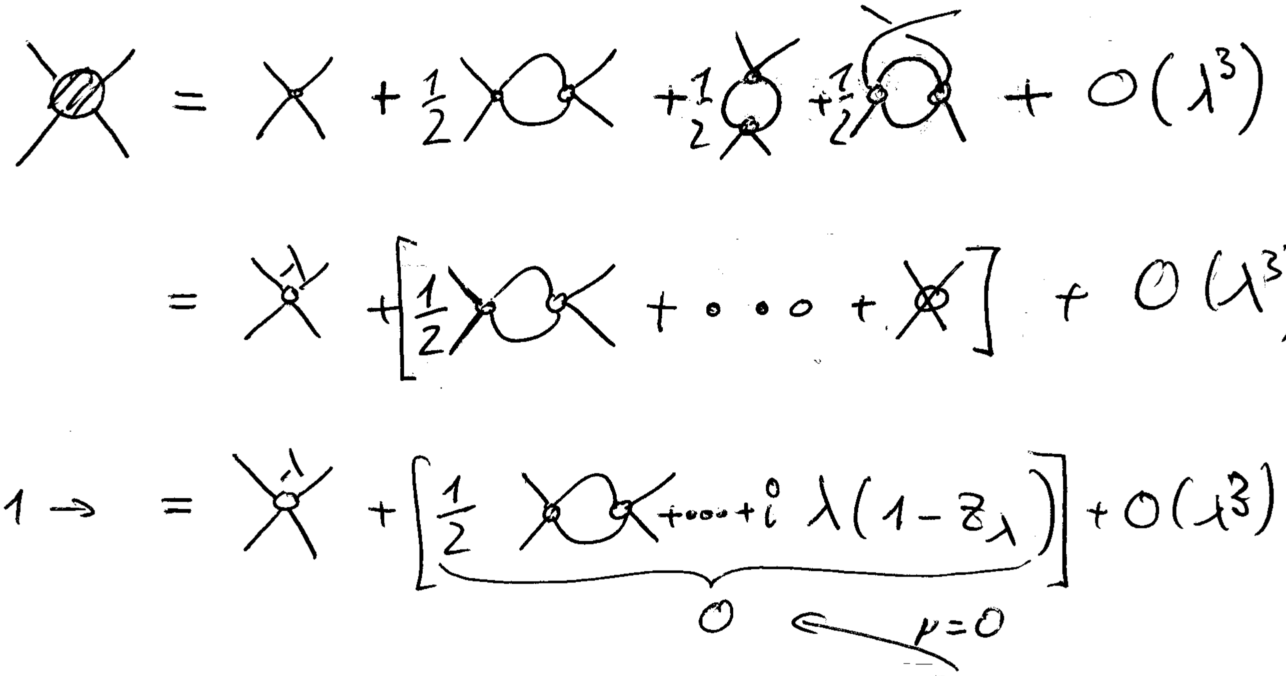
 $$= \frac{i}{p^2 - m^2} + O(\lambda^2) \quad (7.30)$$

Remark: The equivalence of (7.28) and (7.29) is best seen with $\ln\frac{1}{1+\Lambda^2/m^2} = \ln m^2/\Lambda^2 + \ln\frac{1}{1+m^2/\Lambda^2}$

The last term vanishes for $\Lambda \to \infty$.

(2) coupling correction:

$Z_\lambda|_{1\text{-loop}} = 1 \rightarrow$



Renormalisation condition: (for simpl. at $t=s=u=0$)

$\boxed{\mu^2 = 0}$

$$\boxed{1 - Z_\lambda = \frac{3\lambda}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{i}{(q^2 - m^2)^2}} \qquad (7.31)$$

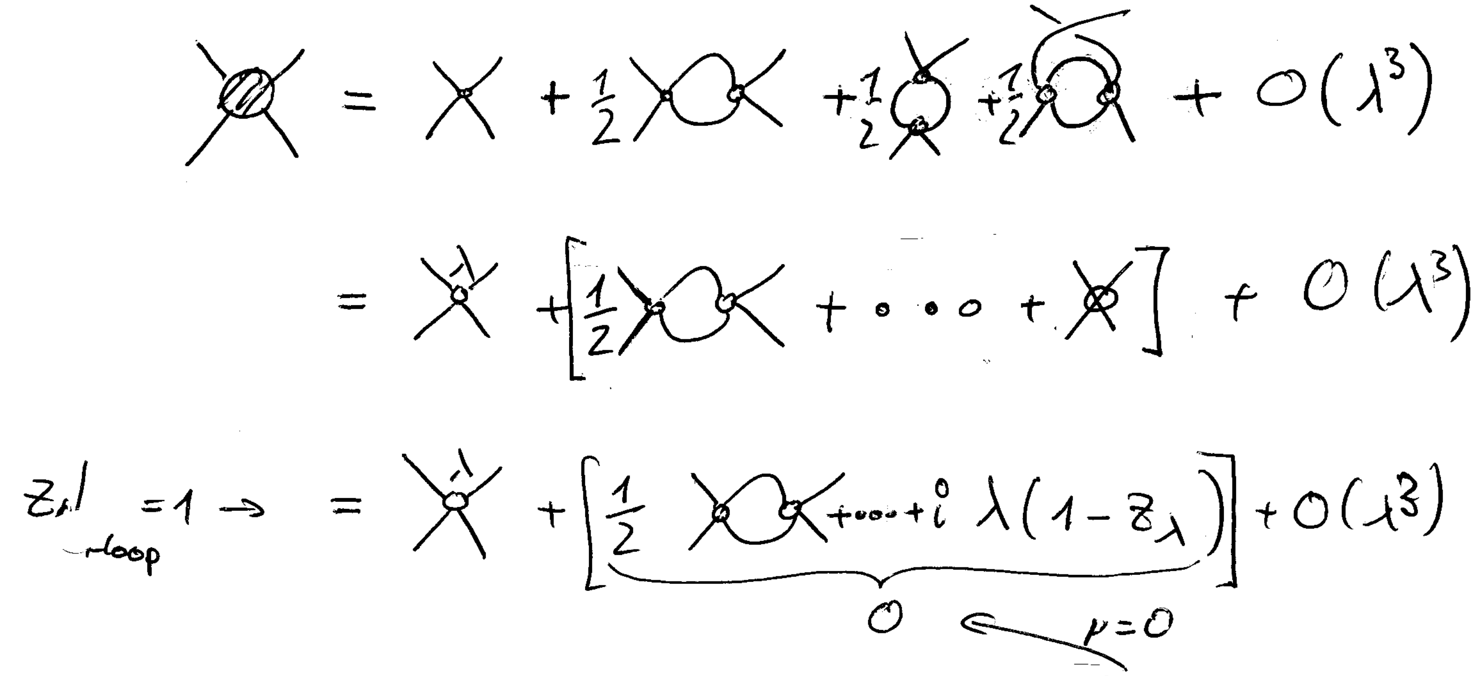
We compute after Wick rotation with dim. reg.:

$$-\frac{3\lambda}{2} \mu^{2\frac{4-d}{2}} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)^2} \overset{\text{p. 187a}}{=} -\frac{3\lambda}{2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\overset{2-d/2}{\varepsilon})}{\Gamma(2)} \left(\frac{m^2}{\mu^2}\right)^{-\varepsilon}$$

$$= -\frac{3\lambda}{2} \frac{1}{16\pi^2} \left[\frac{1}{\varepsilon} - \gamma + \ln 4\pi - \ln m^2/\mu^2 \right].$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon) \qquad (7.32)$$

(2) coupling correction:



Renormalisation condition: (for simpl. at $t=s=u=0$)

$$\boxed{\mu^2=0} \qquad \boxed{1 - Z_\lambda = \frac{3\lambda}{2}\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{(q^2-m^2)^2}} \qquad (7.31)$$

We compute after Wick rotation with dim. reg.:

$$-\frac{3\lambda}{2}\,\mu^{2\frac{4-d}{2}}\int\frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2+m^2)^2} \overset{\text{p. 187a}}{=} -\frac{3\lambda}{2}\,\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(\underbrace{\varepsilon}_{2-d/2})}{\Gamma(2)}\left(\frac{m^2}{\mu^2}\right)^{-\varepsilon}$$

$$= -\frac{3\lambda}{2}\,\frac{1}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma + \ln 4\pi - \ln m^2/\mu^2\right].$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon) \qquad (7.32)$$

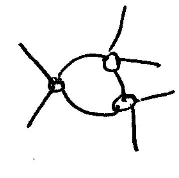
In summary: (at RG-scale $\mu^2 = 0$)

$$\boxed{Z_\lambda = 1 + \frac{3}{2}\,\frac{1}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma + \ln 4\pi - \ln m^2/\mu^2\right]} \tag{7.33}$$

and $Z_\phi = 1$ (p. 185) and Z_m in eq. (7.29).

Remarks:

(i) The renormalised correlation functions $\langle \phi(p_1)\,\phi(p_2)\rangle_{1\text{-loop}}$, $\langle \phi(p_1)\cdots\phi(p_4)\rangle_{1\text{-loop}}$ are <u>finite</u>, but depend on the renormalisation scale μ (scheme dep.)

(ii) Higher correlation functions at one loop are finite from the onset, e.g. $\langle \phi_0(p_1)\cdots\phi_0(p_6)$: 

at $p_i = 0$ $\quad \sim \lambda^3 \int \frac{d^4 q}{(2\pi)^4}\,\frac{1}{(q^2+m^2)^3} \leftarrow$ finite

A singularity in $\langle \phi_0(p_1) \cdots \phi_0(p_6) \rangle$ would be a desaster: there is no counter term!

Perturbative Renormalisability (in ϕ^4-theo.)

$\Longleftrightarrow$ all correlation fct. to <u>all</u> orders in perturbation theory are finite by adjusting Z_ϕ, Z_m, Z_λ.

(iii) Renormalisation group invariance:

'Physics does <u>not</u> depend on renormalisation <u>scheme</u>', i.e.

$$\boxed{\mu \frac{d}{d\mu} \text{Observable} = 0}$$

It also does not depend on the cut-off scale;

$$\boxed{\Lambda \frac{d}{d\Lambda} \text{Obervable} = 0}$$

Evidently, the bare quantities know nothing of the renormalisation point. Hence

$$\mu \frac{d}{d\mu} \phi_0 = \mu \frac{d}{d\mu} m_0 = \mu \frac{d}{d\mu} \lambda_0 = 0 \qquad (7.34)$$

It follows that

$$\mu \frac{d\phi}{d\mu} \frac{1}{\phi} = -\frac{1}{2} \mu \frac{d Z_\phi}{d\mu} \frac{1}{Z_\phi} = -\gamma_\phi$$

beta-functions

$$\mu \frac{d\lambda}{d\mu} \frac{1}{\lambda} = -\mu \frac{d Z_\lambda}{d\mu} \frac{1}{Z_\lambda} = \beta_\lambda$$

$$\mu \frac{d m^2}{d\mu} \frac{1}{m^2} = -\mu \frac{d Z_m}{d\mu} \frac{1}{Z_m} = \gamma_m \qquad (7.35)$$

In turn, the renormalised quantities are insensitive to the cut-off. Hence

$$\Lambda \frac{d}{d\Lambda} \phi = \Lambda \frac{d}{d\Lambda} m = \Lambda \frac{d}{d\Lambda} \lambda = 0 \qquad (7.36)$$

$\Rightarrow$ The Λ and μ scaling are (asymptotically) directly related.

no other scales

(iv) renormalised and running coupling

The renormalised coupling is <u>not</u> the physical coupling, as it runs with μ:

$\mu \frac{d}{d\mu} \ln \lambda = \beta$, see eq. (7.35). In our one loop case we have

$$\boxed{\beta(\mu) = -\mu \frac{d \ln Z_\lambda}{d\mu} = \frac{3}{16\pi^2} \lambda} \qquad (7.37)$$

$$\text{at } \mu = 0 \longrightarrow \; = -m \frac{d \ln Z_\lambda}{dm}$$

Our renormalisation condition, however, fixed $\lambda = \lambda_{phys}$ at the momentum scale μ.

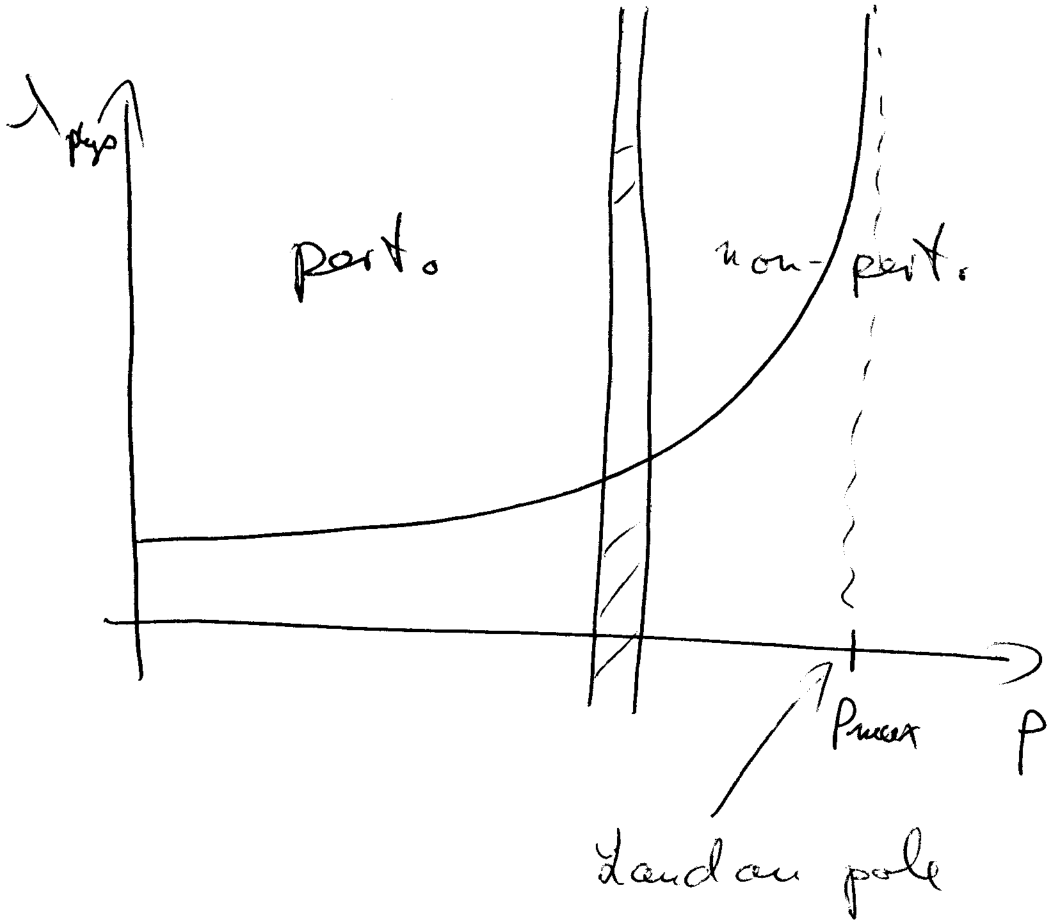
Hence

$$p \frac{d}{dp} \lambda_{phys}(p) \simeq \mu \frac{d}{d\mu} \lambda \Big|_{\mu = p}$$

asymptotically, $p^2 \gg m^2$

We can integrate this eq. at one loop and get

$$\lambda_{phys}(p) = \frac{\lambda_0}{1 + \frac{3\lambda_0}{16\pi^2}\left(\ln p_{max}/p\right)} \qquad (7.38)$$



$p_{max} = \bar{p}\, e^{16\pi^2/3\lambda}$

This is linked to the triviality of ϕ^4-theory: $\lambda_{phys}(p) < \infty \;\; \forall p$

$$\Rightarrow \lambda_{phys} \stackrel{!}{=} 0 \qquad (7.39)$$

Note that this has to be proven <u>non</u>-perturbatively

non-pert. QFT $\Rightarrow$ QFT-<u>II</u>