HEIDELBERG UNIVERSITY

INSTITUTE FOR THEORETICAL PHYSICS

Quantum Field Theory I

LECTURE NOTES FROM 2016-2017

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Contents

Introductory words

Introductory words

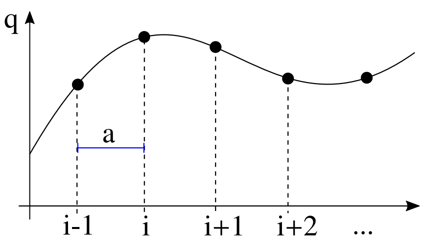
1. Introduction

Quantum Field Theory describes the fundamental interactions of matter.

$$\begin{tikzcd}
x, p & \phi(\mathbf{x}),\ \Pi(\mathbf{y}) \\
\text{classical mechanics} \arrow[r] \arrow[d, "{[\hat{x},\hat{p}]=\mathrm{i}\hbar}"'] & \text{classical field theory} \arrow[d, "{[\hat{\phi}(\mathbf{x}),\hat{\pi}(\mathbf{y})]=\mathrm{i}\hbar\delta(\mathbf{x}-\mathbf{y})}"] \\
\text{quantum mechanics} \arrow[r] & \text{quantum field theory} \\
\hat{x},\ \hat{p} & \hat{\phi}(\mathbf{x}),\ \hat{\pi}(\mathbf{y})
\end{tikzcd}$$

It has various physical applications, such as the standard model, potentially quantum gravity, condensed matter systems and ultracold gases. Further, modern particle physics is also described by quantum field theories: The Higgs-boson corresponds to a scalar field, leptons and quarks are described by fermion fields, and vector fields are used for photons, $W^{\pm}$ and Z bosons, and gluons.

Example 1: Oscillating masses / string.



$$\xrightarrow{a\to 0} \qquad \phi(t,x)$$

Consider the lattice spacing a to become infinitesimal, hence

$$\frac{q_i - q_{i-1}}{a} \quad \to \quad \partial_x \phi$$

and therefore

$$\begin{aligned}
\partial_t^2 q_i &= -c^2 \frac{(q_i - q_{i-1} + q_i - q_{i-1})}{a^2} \\
&\downarrow \\
\partial_t^2 \phi(t,x) &= -c^2 \partial_x^2 \phi(t,x)\,.
\end{aligned}$$

Then for the action it follows,

$$\begin{aligned} S[q] &= \int \mathrm{d}t\, \mathcal{L}(q(t), \dot{q}(t), t) \\ &= a \int \mathrm{d}t \sum_i \frac{(\partial_t q_i)^2 - c^2(q_{i+1} - q_i)^2}{a^2} \\ \downarrow \\ S[\phi] &= \int \mathrm{d}t \int \mathrm{d}x \left((\partial_t \phi)^2 - c^2(\partial_x \phi)^2\right). \end{aligned}$$

In general dimensions the action of a scalar field ϕ can be written as

$$S[\phi] = \int \mathrm{d}^d x \left(\frac{1}{2}(\partial_t \phi)^2 - (\nabla\phi)^2 - V(\phi)\right),$$

where V denotes the potential and $c = 1$.
Thus, the problem can be simply described by a bunch of (coupled) harmonic oscillators.

Example 2: Electrodynamics.

Consider a vector field A_μ with the action

$$S[A_\mu] = \int \mathrm{d}^4 \mathcal{L}(A_\mu(x), \partial_\mu A_\nu(x)),$$

where $x^0 = t$ and $\left(x^i\right) = \mathbf{x}$ for $i = 1, 2, 3$.
The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

with

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\mu - \partial_\nu A_\mu \\ F^{\mu\nu} &= \eta^{\mu\rho}\eta^{\nu\sigma} F_{\rho\sigma} \end{aligned}$$

and the flat metric

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In the following chapter we will discuss free scalar fields. Starting from classical field theory (section I) we move on to symmetries and the Noether theorem (section II). Lastly, we advance from classical theory to quantum field theory through quantisation. This requires the construction of the Fock space (section III), which is basically a sum of a set of Hilbert spaces.

2. Free Scalar Field

I. Classical Theory

At first we consider a real scalar field $\phi(x)$. In particular this means, that ϕ is **invariant under Poincaré transformations** (translations, rotations, boosts), namely

$$\phi(x) \mapsto P(\phi(x)) = \phi(x) \,. \tag{2.1}$$

Poincaré symmetry implies, that the scalar product $(x-y)^2$ remains invariant under the transformation

$$P = (\Lambda, a): \quad x^\mu \mapsto P(x^\mu) \quad = \quad \Lambda^\mu{}_\nu x^\nu + a^\mu \,, \tag{2.2}$$

where Λ denotes the Lorentz transformation with

$$\left(\Lambda^T \eta \Lambda\right) = \eta \,, \tag{2.3}$$

or in components

$$\Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\mu = \eta_{\mu\nu} \,. \tag{2.4}$$

For a composition of Poincaré transformations it holds

$$(\Lambda_1, a_1) \circ (\Lambda_2, a_2) \quad = \quad (\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1) \,. \tag{2.5}$$

Furthermore, for a scalar field ϕ the **action is Lorentz invariant**, which we will show exemplary for the common case, where

$$S[\phi] = \int \mathrm{d}^4 x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \,, \tag{2.6}$$

with

$$\mathcal{L} \quad = \quad \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi) \tag{2.7}$$

$$V(\phi) \quad = \quad \frac{1}{2} m \phi^2 + \mathcal{O}(\phi^3) \,. \tag{2.8}$$

Lorentz invariance follows from

$$\begin{aligned} \partial_\mu \phi \, \partial^\mu \phi \quad &\mapsto \quad \partial_\nu \phi \, \Lambda_\mu{}^\nu \Lambda^\mu{}_\rho \, \partial^\rho \phi \\ &= \quad \partial_\nu \phi \, \left(\Lambda^T \eta \Lambda\right)^\nu{}_\rho \, \partial^\rho \phi \\ &= \quad \partial_\nu \phi \, \partial^\nu \phi \quad \text{(with Eq. (2.3))} \end{aligned} \tag{2.9}$$

$$V(\phi) \quad \mapsto \quad V(\phi) \tag{2.10}$$

$$\Rightarrow \mathcal{L} \quad \mapsto \quad \mathcal{L} \tag{2.11}$$

$$\Rightarrow S \quad \mapsto \quad S \,. \tag{2.12}$$

According to Hamilton's principle, the variation of the action has to be zero. This can be used to derive the **equation of motion** (EoM). For the case above it holds

$$
\begin{aligned}
0 \stackrel{!}{=} \delta S &= \delta \int \mathrm{d}^4x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right) \\
&= \int \mathrm{d}^4x \left(\partial_\mu\phi\,(\partial_\nu\delta\phi)\,\eta^{\mu\nu} - m^2\phi\,\delta\phi\right) \quad \text{as} \quad \delta(\partial\phi)^2 = (\delta\partial_\mu\phi\,\partial_\nu\phi)\,\eta^{\mu\nu} \\
&= -\int \mathrm{d}^4x \left(\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi\right)\delta\phi \quad \text{(using partial integration)} \\
&= -\int \mathrm{d}^4x\,\delta\phi\,(\partial^2 + m^2)\,\phi\,.
\end{aligned}
\tag{2.13}
$$

Therefore, the scalar field must satisfy the

Klein-Gordon equation

$$
(\partial^2 + m^2)\,\phi(x) = 0\,, \tag{2.14}
$$

which is the equation of motion for a four dimensional scalar field. Hence, we obtain an expression for the scalar field ϕ by solving Eq. (2.14). In the following the solution for 1+0 dimensional theory is derived and subsequently generalised. 1+0 dimensional theory describes ordinary (quantum) mechanics, as

$$
\phi(t,\mathbf{x})\Big|_{1+0\ \mathrm{dim}} = \phi(t) = q(t)\,, \tag{2.15}
$$

$$
\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{1}{2}m^2q^2 - \frac{\lambda}{4}q^4\,. \tag{2.16}
$$

The first two terms correspond to a harmonic oscillator and the last is an anharmonic term. The equation of motion is then the Euler-Lagrange equation

$$
\partial_t\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = 0\,. \tag{2.17}
$$

It follows that

$$
\ddot{q} + m^2q + \lambda q^3 = 0\,. \tag{2.18}
$$

For $\lambda = 0$ this is the differential equation of a harmonic oscillator, which is solved by a plane wave

$$
q(t) = A_0\,\mathrm{e}^{ikt} \quad \text{with} \quad k^2 - m^2 = 0\,. \tag{2.19}
$$

When extending to 1+d dimensions, ϕ describes a density of coupled harmonic oscillators with the general solution

$$
\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}}\left(\alpha(\mathbf{k})\,\mathrm{e}^{-ikx} + \alpha^*(\mathbf{k})\,\mathrm{e}^{ikx}\right) \quad \text{with} \quad \omega_{\mathbf{k}} := \sqrt{\mathbf{k}^2 + m^2}\,. \tag{2.20}
$$

Note, that ϕ is real and satisfies Eq. (2.14). Further note, that

$$
\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}(2\pi)\,\delta(k^2 - m^2)\,\theta(k^0)\,, \tag{2.21}
$$

i.e. that this is a Lorentz invariant measure. This can be derived by using

$$\delta(g(x) - g(a)) = \frac{1}{|g'(a)|}\,\delta(x-a)\,, \tag{2.22}$$

where $g(x)$ is any C^1 function. Then

$$\begin{aligned}\delta(k^2 - m^2) &= \delta((k^0)^2 - \mathbf{k}^2 - m^2) = \delta(k_0^2 - \omega_{\mathbf{k}}^2)\\ &= \frac{1}{|2\omega_{\mathbf{k}}|}\,\delta(k_0 - \omega_{\mathbf{k}})\,.\end{aligned} \tag{2.23}$$

It follows

$$\begin{aligned}\int \frac{\mathrm{d}^4k}{(2\pi)^4}(2\pi)\,\delta(k^2 - m^2)\,\theta(k^0) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}(2\pi)\,\frac{1}{|2\omega_{\mathbf{k}}|}\,\delta(k_0 - \omega_{\mathbf{k}})\,\theta(k^0)\\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}}\,.\end{aligned} \tag{2.24}$$

Lastly, let us consider the case of a **complex scalar field**

$$\phi(x) = \frac{1}{\sqrt{2}}\Big(\phi_1(x) + \mathrm{i}\,\phi_2(x)\Big)\,, \tag{2.25}$$

where ϕ_1 and ϕ_2 are both real scalar fields. The action is given by

$$S[\phi] = \int \mathrm{d}^4x\,\mathcal{L}(\phi, \partial_\mu\phi)\,, \tag{2.26}$$

with

$$\begin{aligned}\mathcal{L}(\phi, \partial_\mu\phi) &= \partial_\mu\phi\,\partial^\mu\phi^* - m^2\phi\,\phi^*\\ &= \frac{1}{2}\Big((\partial\phi_1)^2 + (\partial\phi_2)^2 - m^2(\phi_1^2 + \phi_2^2)\Big).\end{aligned} \tag{2.27}$$

Then the general solution of Eq. (2.14) becomes

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}}\Big(\alpha(\mathbf{k})\,\mathrm{e}^{-ikx} + \beta^*(\mathbf{k})\,\mathrm{e}^{ikx}\Big) \quad \text{with} \quad \omega_{\mathbf{k}} := \sqrt{\mathbf{k}^2 + m^2}\,. \tag{2.28}$$

Note, that the action is invariant under multiplication with a global phase $\mathrm{e}^{i\omega}$. This global U(1) symmetry implies a conserved charge, as will be discussed in the subsequent section.

II. Noether Theorem

Noether's theorem states, that every continuous symmetry of the action leads to a conserved current density and a conserved charge. Let us therefore consider a infinitesimal symmetry transformation δ_ϵ such that

$$\phi(x) \mapsto \phi(x) + \delta_\epsilon\phi(x) \tag{2.29}$$

$$S[\phi(x)] \mapsto S[\phi(x) + \delta_\epsilon\phi(x)]\Big|_{\text{EoM}} \overset{!}{=} S[\phi(x)]\,. \tag{2.30}$$

This is satisfied for

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon \partial_\mu J^\mu(\phi)\,, \tag{2.31}$$

as an additional surface term $J(\phi)$ does not affect the action

$$S[\phi] = \int \mathrm{d}^4x\, \mathcal{L} \mapsto \int \mathrm{d}^4x\, \mathcal{L} + \epsilon \int \mathrm{d}^4x\, \partial_\mu J^\mu(\phi)\,, \tag{2.32}$$

because the last term vanishes, if

$$J[\phi]\Big|_{\pm\infty} = 0\,. \tag{2.33}$$

This is true, if $J[\phi]$ is the total derivative of the Lagrangian. Thus, we can write

$$\begin{aligned}
\mathcal{L} \quad &\mapsto \quad \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi}\,\delta_\epsilon\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu\delta_\epsilon\phi \\
&= \quad \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi}\,\delta_\epsilon\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta_\epsilon\phi\right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta_\epsilon\phi \\
&= \quad \mathcal{L} + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta_\epsilon\phi\right) + (\text{EoM})\,\delta_\epsilon\phi \\
&\overset{!}{=} \quad \mathcal{L} + \epsilon\,\partial_\mu J^\mu\,.
\end{aligned} \tag{2.34}$$

Evaluated at the EoM: $\left.\frac{\partial\delta_\epsilon\phi}{\partial\epsilon}\right|_{\epsilon=0} = \Delta\phi$

$$\partial_\mu J^\mu = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\Delta\phi\right)\,. \tag{2.35}$$

We now define the

conserved current (for a single field)

$$j^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\Delta\phi - J^\mu \quad \text{with} \quad \partial_\mu j^\mu = 0\,. \tag{2.36}$$

The conservation law can also be expressed in terms of the

Noether charge

$$Q(t) := \int \mathrm{d}^3x\, j^0(t,\mathbf{x}) \quad \text{with} \quad \dot{Q}(t) = 0\,. \tag{2.37}$$

If the symmetry involves more than one field, the first term in j^μ needs to be replaced by a sum of terms. Consequently, the above extends to a general symmetry with r parameters as: $\Delta_r\phi = \left.\frac{\partial\delta_\epsilon\phi}{\partial\epsilon_r}\right|_{\epsilon=0}$

conserved current (for multiple fields)

$$j^\mu_r := \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\Delta_r\phi_i - J^\mu_r \quad \text{with} \quad \partial_\mu j^\mu = 0\,. \tag{2.38}$$

Note, that splitting $\delta_\epsilon\phi$ into an infinitesimal transformation of ϕ and x, namely

$$\delta_\epsilon\phi = \delta_{\phi_\epsilon}\phi + \delta_\epsilon x_\mu\,\partial^\mu\phi\,, \tag{2.39}$$

leads to: $\Delta x = \dfrac{\partial \delta_\epsilon x}{\partial \epsilon_r}$

$$j_r^\mu = \Delta_r \phi \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \Delta_r x^\nu \left(\partial_\nu \phi \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \mathcal{L} \eta^\mu{}_\nu \right) - J_r^\mu \, . \tag{2.40}$$

As the Physics remains invariant under a shift of the laboratory system, let us now examine the symmetry under **translations**, which corresponds to **energy-momentum conservation**. For this purpose we consider an infinitesimal translation

$$\phi(x) \mapsto \phi(x + \epsilon) = \phi(x) + \epsilon^\mu \partial_\mu \phi(x) + O(\epsilon^2) \, . \tag{2.41}$$

Then the Lagrangian becomes

$$\begin{aligned} \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \quad &\mapsto \quad \mathcal{L} + \epsilon^\mu \partial_\mu \mathcal{L} \\ &= \quad \mathcal{L} + \epsilon^\nu \partial_\mu \eta^\mu{}_\nu \mathcal{L} \, . \end{aligned} \tag{2.42}$$

With $r = \nu$ we obtain

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon^r \partial_\mu J^\mu{}_r \, , \tag{2.43}$$

with

$$J^\mu{}_\nu = \eta^\mu{}_\nu \mathcal{L} \, . \tag{2.44}$$

Then, we define the conserved current as

energy-momentum tensor (or stress-energy tensor)

$$T^\mu{}_\nu := j^\mu{}_\nu = \partial_\nu \phi \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \eta^\mu{}_\nu \mathcal{L} \quad \text{with} \quad \partial_\mu T^{\mu\nu} = 0 \, . \tag{2.45}$$

This indicates, that we have four conserved currents, i.e. charges

$$P^\mu = \int \mathrm{d}^3 x \, T^{0\mu} \, , \tag{2.46}$$

which is referred to as the 4-momentum. The energy-density for time translations is given by the zeroth component of the 4-momentum, namely

$$\begin{aligned} P^0 \quad &= \quad \int \mathrm{d}^3 x \, T^{00} = \int \mathrm{d}^3 x \left((\partial^0 \phi) \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} - \mathcal{L} \right) \\ &= \quad \int \mathrm{d}^3 x \left((\partial^0 \phi) \Pi - \mathcal{L} \right) \\ &= \quad \int \mathrm{d}^3 x \, \mathcal{H} \\ &= \quad H \, , \end{aligned} \tag{2.47}$$

with the momentum

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \, , \tag{2.48}$$

the Hamiltonian density

$$\mathcal{H} = \Pi \, \partial_0 \phi - \mathcal{L} \, , \tag{2.49}$$

and the Hamiltonian H.
For instance, the Hamiltonian density for the common case of a scalar field with $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$ is

$$\mathcal{H} = \frac{1}{2}\Pi(x)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\,. \tag{2.50}$$

Note, that the covariance of P^μ is not apparent, but that $\int d^3x\, \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} \sim d^3x\, dx^0$ is Lorentz invariant, as it transforms as a scalar. Further note, that P^i generates translations:

$$P^i = \int d^3x\, T^{0i} = \int d^3x\, \Pi\partial^i\phi\,, \tag{2.51}$$

which yields for the real scalar field from Eq. (2.7)

$$P^i = \left(\int d^3x\, \Pi\, \nabla\phi\right)^i\,. \tag{2.52}$$

On the other hand the

Poisson bracket

$$\begin{aligned} \left\{P^i, \phi(\mathbf{x})\right\} &= -\nabla\phi \\ \text{with}\quad \{\phi(\mathbf{x}), \Pi(\mathbf{y})\} &= \delta(\mathbf{x}-\mathbf{y}) \\ \text{and}\quad \{\phi(\mathbf{x}), \phi(\mathbf{y})\} &= 0\,. \end{aligned} \tag{2.53}$$

degenerates translations. As will be shown in the next section, quantisation promotes the Poisson brackets to commutators with operators as arguments.
Further it is remarked, that the canonical energy-momentum tensor is in general not symmetric, i.e. $T^{\mu\nu} \neq T^{\nu\mu}$, due to $\partial_\mu \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}$. However, it can always be symmetrised, which is an important property for the coupling to gravity. An alternative definition results from the variation of the action with respect to the metric $g^{\mu\nu}$:

$$T^{\mu\nu}_{sym} = \frac{1}{\sqrt{-\det g}} \frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g=\eta}\,. \tag{2.54}$$

Note, that

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x-y) \tag{2.55}$$

and that the full variation of the scalar field is Lorentz invariant, i.e. $\phi'(x') = \phi(x)$ (so far we only used $\phi(x) \mapsto \phi(x')$). Then:

$$\begin{aligned} \Delta\phi &= 0 \\ \Delta_\rho x^\nu &= \eta_\rho^{\ \nu} \\ J^\mu_{\ \rho} &= 0 \quad (\mathcal{L}' = \mathcal{L})\,. \end{aligned} \tag{2.56}$$

To conclude this section, the **charge of a complex scalar field** shall be discussed. Therefore we use the field from Eq. (2.25), with the Lagrangian

$$\mathcal{L} = \partial_\mu\phi\, \partial^\mu\phi^* - m^2\phi\,\phi^*\,. \tag{2.57}$$

The action is invariant under a multiplication of ϕ with a global phase (U(1) symmetry):

$$\begin{aligned}
\phi &\mapsto \phi' = \mathrm{e}^{i\epsilon}\phi = \phi + i\epsilon\,\phi + ... \\
\phi^* &\mapsto \phi^{*\prime} = \phi^*\mathrm{e}^{-i\epsilon} = \phi^* - i\epsilon\,\phi^* + ... \\
\mathcal{L} &\mapsto \mathcal{L}' = \mathcal{L} \\
&\Rightarrow J^\mu = 0\,.
\end{aligned} \tag{2.58}$$

With $\Delta\phi = i\phi$ and $\Delta\phi^* = -i\phi$ we obtain the Noether current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\,\Delta\phi^* \tag{2.59}$$

$$\begin{aligned}
&= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\mathrm{i}\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\,\mathrm{i}\phi^* \\
&= \mathrm{i}\Big(\,(\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^*\Big)
\end{aligned} \tag{2.60}$$

and the

Noether charge of a complex scalar field

$$Q = \int \mathrm{d}^3x\; j^0 = i\int \mathrm{d}^3x\,\Big(\phi^*\partial_t\,\phi - (\partial_t\phi^*)\,\phi\Big). \tag{2.61}$$

As the Noether charge is conserved for fields that satisfy the equation of motion, it follows

$$\begin{aligned}
\dot{Q}\Big|_{\mathrm{EoM}} &= \mathrm{i}\int \mathrm{d}^3x\,\Big(\dot\phi^*\,\dot\phi - \dot\phi^*\,\dot\phi + \phi^*\,\partial_t^2\phi - (\partial_t^2\phi^*)\,\phi\Big) \\
\text{using Eq. (2.14)} \rightarrow \quad &= \mathrm{i}\int \mathrm{d}^3x\,\Big(\phi^*(\nabla^2 - m^2)\,\phi - (\nabla^2 - m^2)\,\phi^*\,\phi\Big) \\
&= \mathrm{i}\int \mathrm{d}^3x\,\Big(\phi^*\,\Delta\phi - \Delta\phi^*\,\phi\Big) \\
&= 0 \quad \text{(using partial integration; no boundary terms).}
\end{aligned} \tag{2.62}$$

Or in momentum space (with Eq. (2.28))

$$Q = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\Big(\alpha^*(\mathbf{p})\,\alpha(\mathbf{p}) - \beta^*(\mathbf{p})\beta(\mathbf{p})\Big). \tag{2.63}$$

In the next section we advance by performing quantisation. Then α, α^* and β, β^* will become annihilation and creation operators for particles and anti-particles, respectively.

III. Quantisation

Quantum field theory is the field-theoretical limit of quantum mechanics. Therefore

$$\begin{aligned} [\hat{q},\hat{p}] &= i\hbar \\ [\hat{q},\hat{q}] &= 0 = [\hat{p},\hat{p}] \end{aligned} \quad \longrightarrow \quad \begin{aligned} \left[\hat{\phi}(\mathbf{x}),\hat{\Pi}(\mathbf{y})\right] &= i\,\delta(\mathbf{x}-\mathbf{y}) \\ \left[\hat{\phi}(\mathbf{x}),\hat{\phi}(\mathbf{y})\right] &= 0 = \left[\hat{\Pi}(\mathbf{x}),\hat{\Pi}(\mathbf{y})\right], \end{aligned}$$

where $\hbar = 1$ and $c = 1$ on the right hand side. Note, that the expectation value $\left\langle \hat{\phi}(\mathbf{x}) \right\rangle$ needs to yield the classical field. Again, we start from 1+0 dimensional theory and generalise subsequently. It is

$$\begin{aligned} S[q] &= \int \mathrm{d}t\, \mathcal{L} \\ &= \int \mathrm{d}t \left(\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2\right) \qquad (2.64) \\ H &= p\dot{q} - \mathcal{L} \quad \text{with} \quad p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q} \\ \Rightarrow H &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2\,. \qquad (2.65) \end{aligned}$$

Now we perform quantisation, i.e. $p, q \to \hat{p}, \hat{q}$, with $[\hat{q}, \hat{p}] \coloneqq i\hbar$. We introduce creation and annihilation operator as

$$\hat{q} = \frac{1}{\sqrt{2\omega}}\,(a + a^\dagger) \qquad (2.66)$$

$$\hat{p} = -i\sqrt{\frac{\omega}{2}}\,(a - a^\dagger)\,, \qquad (2.67)$$

with

$$\begin{aligned} [a, a]^\dagger &= 1 \\ [a, a] = \left[a^\dagger, a^\dagger\right] &= 0\,. \end{aligned} \qquad (2.68)$$

The Hamilton operator can than be written in terms of creation and annihilation operator

$$\hat{H} = \left(a^\dagger\, a + \frac{1}{2}\right)\omega\,, \qquad (2.69)$$

where $\frac{1}{2}\,\omega$ corresponds to the vacuum energy. In the Heisenberg picture the operators evolve with time, whereas the states are stationary, i.e.

$$\begin{aligned} i\frac{\partial}{\partial t}\hat{O}(t) &= \left[\hat{O}(t), \hat{H}\right] \\ \hat{O}(t) &= e^{i\hat{H}t}\,\hat{O}(0)\,e^{-i\hat{H}t}\,. \end{aligned} \qquad (2.70)$$

It follows that

$$[\hat{q}(t), \hat{p}(t)] = e^{i\hat{H}t}\,[\hat{q}, \hat{p}]\,e^{-i\hat{H}t} = i\,. \qquad (2.71)$$

For convenience we will now drop the hat marking the operators, as we advance to 1+3 dimensional theory.

The Hamiltonian density for a **real scalar field** is

$$\begin{aligned}\mathcal{H} &= \Pi\,\partial_0\phi - \mathcal{L} \\ &= \frac{1}{2}\Big(\Pi(t,\mathbf{x})^2 + \phi(t,\mathbf{x})(-\Delta + m^2)\phi(t,\mathbf{x})\Big), \end{aligned} \tag{2.72}$$

with $\Delta = \nabla^2$ and

$$\Pi(t,\mathbf{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}(t,\mathbf{x}) = \partial^0\phi = \dot{\phi} \tag{2.73}$$

$$\mathcal{L} = \frac{1}{2}\Big(\partial_0\phi\,\partial^0\phi - (\nabla\phi)^2 - m^2\phi^2\Big). \tag{2.74}$$

Analogue to the 1+0 dimensional theory we obtain the

canonical commutation relations

$$\begin{aligned}[\phi(t,\mathbf{x}),\Pi(t,\mathbf{y})] &= \mathrm{i}\,\delta(\mathbf{x}-\mathbf{y}) \\ [\phi(t,\mathbf{x}),\phi(t,\mathbf{y})] &= [\Pi(t,\mathbf{x}),\Pi(t,\mathbf{y})] = 0\,. \end{aligned} \tag{2.75}$$

The field operator ϕ satisfies the EoM, as does its expectation value $\langle\phi\rangle$. Note, that the free field theory describes a (coupled) set of harmonic oscillators due to $\phi\Delta\phi$. In Fourier space this term becomes $\phi(-p)\,\mathbf{p}^2\,\phi(p)$. Consequently, we can **diagonalise $\mathcal{L}$ and $\mathcal{H}$ in momentum space**. For this purpose we introduce creation and annihilation operator likewise to 1+0 dimensional theory:

$$\begin{aligned}\phi(\mathbf{x}) &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(a(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}px} + a^\dagger(\mathbf{p})\,\mathrm{e}^{\mathrm{i}px}\Big) \\ \Pi(\mathbf{x}) &= \partial^0\phi(x) = -\mathrm{i}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big(a(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}px} + a^\dagger(\mathbf{p})\,\mathrm{e}^{\mathrm{i}px}\Big), \end{aligned} \tag{2.76}$$

with

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}\,. \tag{2.77}$$

The Fourier transform is defined as

$$\begin{aligned}\tilde{\phi}(p) &:= \int\mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}px}\phi(x) \\ \phi(x) &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}px}\tilde{\phi}(p)\,. \end{aligned} \tag{2.78}$$

With the spatial Fourier transform ($t = 0$) we get the representation of the

field operator in momentum space

$$\begin{aligned}\tilde{\phi}(\mathbf{p}) &:= \int d^3x\, e^{-ipx}\, \phi(\mathbf{x}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a(\mathbf{p}) + a^\dagger(-\mathbf{p})\right) \\ \tilde{\Pi}(\mathbf{p}) &:= \int d^3x\, e^{-ipx}\, \partial^0\phi(\mathbf{x}) = -i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(a(\mathbf{p}) - a^\dagger(-\mathbf{p})\right),\end{aligned} \tag{2.79}$$

with the canonical commutation relations

$$\begin{aligned}\left[\tilde{\phi}(\mathbf{p}), \tilde{\Pi}(\mathbf{q})\right] &= \int d^3x\, d^3y\, e^{-i(\mathbf{px}+\mathbf{qy})}\, [\phi(\mathbf{x}), \Pi(\mathbf{y})] \\ &= \int d^3x\, d^3y\, e^{-i(\mathbf{px}+\mathbf{qy})}\, i\,\delta(\mathbf{x}-\mathbf{y}) = i\int d^3x\, e^{i(\mathbf{p}+\mathbf{q})\mathbf{x}} \\ &= i\,(2\pi)^3\,\delta(\mathbf{p}+\mathbf{q})\end{aligned} \tag{2.80}$$

$$\left[\tilde{\phi}(\mathbf{p}), \tilde{\phi}(\mathbf{q})\right] = 0 = \left[\tilde{\Pi}(\mathbf{p}), \tilde{\Pi}(\mathbf{q})\right]. \tag{2.81}$$

That is, $\tilde{\Pi}(\mathbf{q})$ is conjugate to $\tilde{\phi}(-\mathbf{q})$. Hence, we obtain the formulation for

creation and annihilation operator

$$\begin{aligned}a(\mathbf{p}) &= \sqrt{\frac{\omega_{\mathbf{p}}}{2}}\,\tilde{\phi}(\mathbf{p}) + i\,\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\,\tilde{\Pi}(\mathbf{p}) \\ a^\dagger(-\mathbf{p}) &= \sqrt{\frac{\omega_{\mathbf{p}}}{2}}\,\tilde{\phi}(\mathbf{p}) - i\,\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\,\tilde{\Pi}(\mathbf{p}).\end{aligned} \tag{2.82}$$

Note, that this is completely analogue to the harmonic oscillator in quantum mechanics. The creation and annihilation operators obey canonical commutation relations of a density of harmonic oscillators:

$$\begin{aligned}\left[a(\mathbf{p}), a^\dagger(\mathbf{q})\right] &= -\frac{i}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\left[\tilde{\phi}(\mathbf{p}), \tilde{\Pi}(-\mathbf{q})\right] - \frac{i}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\left[\tilde{\phi}(-\mathbf{q}), \tilde{\Pi}(\mathbf{p})\right] \\ \text{using Eq. (2.80)} \quad \rightarrow \quad &= (2\pi)^3\,\delta(\mathbf{p}-\mathbf{q}) \\ [a(\mathbf{p}), a(\mathbf{q})] &= 0 = \left[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{q})\right].\end{aligned} \tag{2.83}$$

Now we can diagonalise the Hamiltonian density in momentum space:

$$\begin{aligned}-\int d^3\phi(\mathbf{x})\,\Delta\phi(\mathbf{x}) &= \int d^3x \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\, e^{-i\mathbf{x}(\mathbf{p}+\mathbf{q})}\,\tilde{\phi}(\mathbf{p})\,\mathbf{q}^2\,\tilde{\phi}(\mathbf{q}) \quad \left(\text{as} \quad \Delta e^{-i\mathbf{qx}} = -\mathbf{q}^2 e^{-i\mathbf{qx}}\right) \\ &= \int \frac{d^3p}{(2\pi)^3}\tilde{\phi}(\mathbf{p})\,\mathbf{p}^2\,\tilde{\phi}(-\mathbf{p}),.\end{aligned} \tag{2.84}$$

Analogously we get

$$
\begin{aligned}
m^2 \int \mathrm{d}^3x\, \phi^2(\mathbf{x}) &= m^2 \int \frac{\mathrm{d}^3p}{(2\pi)^3} \tilde{\phi}(\mathbf{p})\, \tilde{\phi}(-\mathbf{p}) \\
\int \mathrm{d}^3x\, \Pi^2(\mathbf{x}) &= \int \mathrm{d}^3p\, \tilde{\Pi}(\mathbf{p})\, \tilde{\Pi}(-\mathbf{p})\,. \qquad (2.85)
\end{aligned}
$$

This yields the

diagonal Hamiltonian

$$
H = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2} \Big(\tilde{\Pi}(\mathbf{p})\, \tilde{\Pi}(-\mathbf{p}) + \omega_{\mathbf{p}}^2\, \tilde{\phi}(\mathbf{p})\, \tilde{\phi}(-\mathbf{p}) \Big) \quad \text{with} \quad \omega_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2\,. \qquad (2.86)
$$

The Physics interpretation of H is best done in terms of $a,\ a^\dagger$. Hence, we use Eq. (2.79) to rewrite

$$
\begin{aligned}
H &= \frac{1}{2} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \omega_{\mathbf{p}} \Big[a^\dagger(\mathbf{p})\, a(\mathbf{p}) + \frac{1}{2}\Big(a^\dagger(\mathbf{p})\, a^\dagger(-\mathbf{p}) + a(\mathbf{p})\, a(-\mathbf{p}) \Big) + \ldots \\
&\quad \ldots + a^\dagger(\mathbf{p})\, a(\mathbf{p}) - \frac{1}{2}\Big(a^\dagger(\mathbf{p})\, a^\dagger(-\mathbf{p}) + a(\mathbf{p})\, a(-\mathbf{p}) \Big) + \frac{1}{2} \big[a(\mathbf{p}), a^\dagger(\mathbf{p}) \big] \Big] \\
\text{using Eq. (2.83)} \quad \rightarrow \quad &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \omega_{\mathbf{p}}\, a^\dagger(\mathbf{p})\, a(\mathbf{p}) + \frac{1}{2} V \int \frac{\mathrm{d}^3p}{(2\pi)^3} \omega_{\mathbf{p}}\,, \qquad (2.87)
\end{aligned}
$$

with

$$
\big[a(\mathbf{p}), a^\dagger(\mathbf{p}) \big] = (2\pi)^3 \delta(\mathbf{0}) = \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}} \Big|_{\mathbf{p}=0} = V\,, \qquad (2.88)
$$

where V is the volume of $\mathbb{R}^3$.
The second term in Eq. (2.87) comprises **two infinities**. Firstly, the volume of $\mathbb{R}^3$, which is referred to as "infra-red infinity". This can be dealt with, by putting the theory in a finite volume, i.e. a box. Then the volume factor is finite and one advances by taking the limit for $V \to \infty$. The second infinity is given by the vacuum energy density, as $\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_{\mathbf{p}}$ diverges. This is called "ultraviolet infinity". We justify to drop this term, by arguing that no absolute energy can be defined, and only energy *differences* are important. It does, however, play a role at finite temperature or when regarding QFT coupled to gravity ("cosmological constant problem") and in QFT with boundary conditions (Casimir effect in QED: attractive force between conducting plates). Finally, we have the

Hamiltonian

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_{\mathbf{p}}\, a^\dagger(\mathbf{p})\, a(\mathbf{p}) \quad \text{with} \quad \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}\,. \tag{2.89}$$

H is the Hamiltonian of a momentum continuum of harmonic oscillators with frequencies $\omega_{\mathbf{p}}$. The interpretation of $a,\ a^\dagger$ is that of annihilation and creation operator, respectively.
Next, we construct the **Fock space**, which is basically a sum of a set of Hilbert spaces. We normalise energy differences from the vacuum/ground state to zero, i.e. we define the

vacuum state

$$H\,|0\rangle = 0 \quad \text{with} \quad a(\mathbf{p})\,|0\rangle = 0 \quad \text{and} \quad \langle 0|0\rangle = 1\,. \tag{2.90}$$

All states are generated by applying $a,\ a^\dagger$ on the vacuum state $|0\rangle$. Therefore,

$$\begin{aligned} H\,|\mathbf{p}\rangle &= \int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'}\, a^\dagger(\mathbf{p}')\, a(\mathbf{p}')\, \sqrt{2\omega_{\mathbf{p}}}\, a^\dagger(\mathbf{p})\,|0\rangle \\ &= \int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'}\, a^\dagger(\mathbf{p}')\, \sqrt{2\omega_{\mathbf{p}}} \Big(\big[a(\mathbf{p}'), a^\dagger(\mathbf{p})\big] + a^\dagger(\mathbf{p})\, a(\mathbf{p}') \Big) |0\rangle \\ &= \omega_{\mathbf{p}}\, a^\dagger(\mathbf{p})\, \sqrt{2\omega_{\mathbf{p}}}\,|0\rangle = \omega_{\mathbf{p}}\,|\mathbf{p}\rangle\;. \end{aligned} \tag{2.91}$$

We obtain the

one-particle state (with specific momentum)

$$H\,|\mathbf{p}\rangle = \omega_{\mathbf{p}}\,|\mathbf{p}\rangle \quad \text{with} \quad |\mathbf{p}\rangle = \sqrt{2\omega_{\mathbf{p}}}\, a^\dagger(\mathbf{p})\,|0\rangle\;. \tag{2.92}$$

Note, that

$$\begin{aligned} \langle \mathbf{p}' \big| \mathbf{p} \rangle &= 2\sqrt{\omega_{\mathbf{p}}\,\omega_{\mathbf{p}'}}\, \langle 0|\, a(\mathbf{p}')\, a^\dagger(\mathbf{p})\,|0\rangle \\ &= 2\sqrt{\omega_{\mathbf{p}}\,\omega_{\mathbf{p}'}}\, \langle 0|\, \big[a(\mathbf{p}'), a^\dagger(\mathbf{p})\big]\,|0\rangle \\ &= 2\omega_{\mathbf{p}}\,(2\pi)^3\,\delta(\mathbf{p} - \mathbf{p}')\,, \end{aligned} \tag{2.93}$$

which implies, that one-particle states with different momenta are orthogonal. Now we include all different momenta that the state can have, to derive the

general one-particle state

$$|f\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \mathrm{f}(\mathbf{p})\, a^\dagger(\mathbf{p})\, |0\rangle \ , \tag{2.94}$$

where $\mathrm{f}(\mathbf{p})$ denotes the amplitude, i.e. the distribution of momenta present in the state. It is

$$\begin{aligned} H\,|f\rangle &= \int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'}\, a^\dagger(\mathbf{p}')\, a(\mathbf{p}') \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \mathrm{f}(\mathbf{p})\, a^\dagger(\mathbf{p})\, |0\rangle \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}}\, a^\dagger(\mathbf{p})\, \mathrm{f}(\mathbf{p})\, |0\rangle \\ \langle f|f\rangle &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}} 2\omega_{\mathbf{q}}}}\, \mathrm{f}^*(\mathbf{p})\, \mathrm{f}(\mathbf{q})\, \langle 0|\, a(\mathbf{p})\, a^\dagger(\mathbf{q})\, |0\rangle \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}}\, \mathrm{f}^*(\mathbf{p})\, \mathrm{f}(\mathbf{p})\, . \end{aligned} \tag{2.95}$$

Let us now extend this to the N-particle state:

$$H\,|\mathbf{p_1} \cdots \mathbf{p_N}\rangle = \int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'}\, a^\dagger(\mathbf{p}')\, a(\mathbf{p}')\, 2^{N/2} \sqrt{\omega_{\mathbf{p_1}} \cdots \omega_{\mathbf{p_N}}}\, a^\dagger(\mathbf{p_1}) \cdots a^\dagger(\mathbf{p_N})\, |0\rangle \ . \tag{2.96}$$

In the next step we shift all creation operators to the left and all annihilation operators to the right, which is called **normal ordering**. Then we use Eq. (2.90) and the commutation relations to simplify the equation. Using

$$\begin{aligned} a(\mathbf{p}') \left(a^\dagger(\mathbf{p_1}) \cdots a^\dagger(\mathbf{p_N}) \right) |0\rangle &= \left(\left[a(\mathbf{p}'), a^\dagger(\mathbf{p_1}) \right] a^\dagger(\mathbf{p_2}) \cdots a^\dagger(\mathbf{p_N}) + \ldots \right. \\ &\quad \ldots + a^\dagger(\mathbf{p_1}) \left[a(\mathbf{p}'), a^\dagger(\mathbf{p_2}) \right] a^\dagger(\mathbf{p_3}) \cdots a^\dagger(\mathbf{p_N}) + \ldots + \ldots \\ &\quad \left. \ldots + a^\dagger(\mathbf{p_1}) \cdots a^\dagger(\mathbf{p_{N-1}}) \left[a(\mathbf{p}'), a^\dagger(\mathbf{N}) \right] \right) |0\rangle \ , \end{aligned} \tag{2.97}$$

this leads to the

N-particle state

$$H\,|\mathbf{p_1} \cdots \mathbf{p_N}\rangle = \left(\sum_{i=1}^{N} \omega_{\mathbf{p_i}} \right) |\mathbf{p_1} \cdots \mathbf{p_N}\rangle \ . \tag{2.98}$$

Note, that the states have Bose symmetry, i.e. that

$$|\mathbf{p_1} \cdots \mathbf{p_i}\, \mathbf{p_{i+1}} \cdots \mathbf{p_N}\rangle = |\mathbf{p_1} \cdots \mathbf{p_{i+1}}\, \mathbf{p_i} \cdots \mathbf{p_N}\rangle \ , \tag{2.99}$$

as $\left[a^\dagger(\mathbf{p_i}), a^\dagger(\mathbf{p_{i+1}}) \right] = 0$. Further note, that the particle states are eigenstates of the Hamiltonian and that the energy-momentum is additive. Therefore, if we have a state $|\beta\rangle$ with $H\,|\beta\rangle = E_\beta\,|\beta\rangle$, $a^\dagger(\mathbf{p})\,|\beta\rangle$ is a state with one additional particle with momentum $\mathbf{p}$ and

$$\begin{aligned} H\Big(a^\dagger(\mathbf{p})\,|\beta\rangle\Big) &= a^\dagger(\mathbf{p})\,H\,|\beta\rangle + \Big[H, a^\dagger(\mathbf{p})\Big]\,|\beta\rangle \\ &= (H+\omega_{\mathbf{p}})\Big(a^\dagger(\mathbf{p})\,|\beta\rangle\Big). \end{aligned} \tag{2.100}$$

Example 3: Annihilation operator applied to general one particle state.

$$\begin{aligned} a(\mathbf{p})\,|f\rangle &= a(\mathbf{p})\int \frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}'}}}\mathrm{f}(\mathbf{p}')\,a^\dagger(\mathbf{p}')\,|0\rangle \\ &= \int \frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}'}}}\mathrm{f}(\mathbf{p}')\,\Big[a(\mathbf{p}), a^\dagger(\mathbf{p}')\Big]\,|0\rangle \\ &= \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\mathrm{f}(\mathbf{p})\,|0\rangle \end{aligned} \tag{2.101}$$

Knowing all this, the **interpretation of the field operator** $\phi(x)$ (Eq. (2.76)) is, that it creates and annihilates a particle at position x. Therefore, states with defined particle number have a vanishing expectation value of ϕ, e.g.

$$\langle 0|\phi(\mathbf{x})|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\,\langle 0|a(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}\mathbf{p}\mathbf{x}} + a^\dagger(\mathbf{p})\,\mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}}|0\rangle = 0\,, \tag{2.102}$$

with

$$\langle 0|a|0\rangle = 0 = \langle 0|a^\dagger|0\rangle\;. \tag{2.103}$$

In the same manner it follows, that

$$\begin{aligned} \langle\mathbf{p}|\phi(\mathbf{x})|\mathbf{p}\rangle &= 0 \\ &\;\vdots \\ \langle\mathbf{p_1}\cdots\mathbf{p_N}|\phi(\mathbf{x})|\mathbf{p_1}\cdots\mathbf{p_N}\rangle &= 0\,, \end{aligned} \tag{2.104}$$

by using

$$|\mathbf{p_1}\cdots\mathbf{p_N}\rangle \simeq a^\dagger(\mathbf{p_1})\cdots a^\dagger(\mathbf{p_N})\,|0\rangle\;. \tag{2.105}$$

Further, applying ϕ to the vacuum state yields

$$\begin{aligned} \phi(\mathbf{x})\,|0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(a(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}\mathbf{p}\mathbf{x}} + a^\dagger(\mathbf{p})\,\mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}}\Big)|0\rangle \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}}\,|\mathbf{p}\rangle\;, \end{aligned} \tag{2.106}$$

which is basically a particle and a plane wave. Thus,

$$\langle 0|\,\phi(\mathbf{x})\,|\mathbf{p}\rangle = \mathrm{e}^{-\mathrm{i}\mathbf{p}\mathbf{x}}\,, \tag{2.107}$$

is a plane wave travelling at momentum $\mathbf{p}$ and reminiscent of non-relativistic QM, as $\langle x|p\rangle = \mathrm{e}^{-\mathrm{i}px}$.
As stated before, classical theory should result from the expectation value. For this purpose we introduce the **coherent state** $|\alpha\rangle$ as

$$a(\mathbf{p})\,|\alpha\rangle = \alpha(\mathbf{p})\,|\alpha\rangle\ , \tag{2.108}$$

with

$$\langle\alpha|\phi(\mathbf{x})|\alpha\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(\mathrm{e}^{-\mathrm{i}\mathbf{px}}\,\alpha(\mathbf{p}) + \mathrm{e}^{\mathrm{i}\mathbf{px}}\,\alpha^*(\mathbf{p})\right). \tag{2.109}$$

Note, that here $\alpha(\mathbf{p})$, $\alpha^*(\mathbf{p})$ are no operators, and Eq. (2.109) is equivalent to Eq. (2.20), i.e. the classical real scalar field. As $|\alpha\rangle$ is a eigenstate of the annihilation operator, it remains unchanged by annihilation (detection) of a particle with momentum $\mathbf{p}$. Hence, it must be a superposition of one-particle states. We make the following ansatz

$$|\alpha\rangle = \frac{1}{\mathcal{N}(\alpha)}\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{\infty}\left(\int\frac{\mathrm{d}^3p_i}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p_i}}}}\,\alpha(\mathbf{p_i})\right)|\mathbf{p_1}\cdots\mathbf{p_n}\rangle\ , \tag{2.110}$$

with

$$a(\mathbf{p})\,|\mathbf{p_1}\cdots\mathbf{p_n}\rangle = \sum_{i=1}^{n}(2\pi)^3\,\sqrt{2\omega_{\mathbf{p_i}}}\,|\mathbf{p_1}\cdots\mathbf{p_{i-1}}\,\mathbf{p_{i+1}}\cdots\mathbf{p_n}\rangle\,\delta(\mathbf{p}-\mathbf{p_i})\,, \tag{2.111}$$

and the normalisation $\mathcal{N}(\alpha)$ such that

$$\langle\alpha|\alpha\rangle = 1\,. \tag{2.112}$$

The explicit calculation of $\langle\alpha|\alpha\rangle$ is quite technical and provided in appendix I. As result, we get

$$\mathcal{N}(\alpha) = \exp\left(\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,|\alpha(\mathbf{p})|^2\right). \tag{2.113}$$

Finally, we get the

coherent state

$$|\alpha\rangle = \frac{1}{\mathcal{N}(\alpha)}\exp\left(\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\alpha(\mathbf{p})\,a^\dagger(\mathbf{p})\right)|0\rangle\ . \tag{2.114}$$

Note, that the scalar product is

$$\begin{aligned}\langle\alpha'|\alpha\rangle &= \exp\left(-\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left(|\alpha(\mathbf{p})|^2 + |\alpha'(\mathbf{p})|^2 - 2\alpha'^*\,\alpha(\mathbf{p})\right)\right.\\ &= \frac{1}{\mathcal{N}(\alpha)\,\mathcal{N}(\alpha')}\,,\end{aligned} \tag{2.115}$$

where we used a special case of the Baker–Campbell–Hausdorff formula

$$\exp(A)\,\exp(B) = \exp(B)\exp(A)\exp([A,B]) \quad \text{for} \quad [A,[A,B]] = 0 = [B,[B,A]]\ . \tag{2.116}$$

This implies, that coherent states are not orthogonal. For completeness it is remarked, that in quantum mechanics (1+0 dimensional theory) it is

$$\frac{1}{\Pi}\int\mathrm{d}^2\alpha\ |\alpha\rangle\langle\alpha| = \mathbb{1}\,. \tag{2.117}$$

Quantisation also applies to the conserved **energy-momentum tensor**. Corresponding to Eq. (2.46) we calculate the spatial momentum operator as

$$\begin{aligned}
\mathrm{P}^i &= \int \mathrm{d}^3x\, T^{0i} = \int \mathrm{d}^3x\, \Pi\, \partial^i\phi \\
&= \int \mathrm{d}^3x\, (-\mathrm{i}) \int \frac{\mathrm{d}^3p'}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}'}}{2}} \Big(a(\mathbf{p}')\, \mathrm{e}^{-\mathrm{i}\mathbf{p}'\mathbf{x}} - a^\dagger(\mathbf{p}')\, \mathrm{e}^{\mathrm{i}\mathbf{p}'\mathbf{x}}\Big) + \dots \\
&\quad \dots + \int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} \Big(a(\mathbf{q})\, (-\mathrm{i}q^i)\, \mathrm{e}^{-\mathrm{i}\mathbf{q}\mathbf{x}} + a^\dagger(\mathbf{q})\, \mathrm{i}q^i\, \mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{x}}\Big) \\
&= -\frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \Big(a(\mathbf{p})\, a(-\mathbf{p})\, \mathrm{i}\mathbf{p} + a(\mathbf{p})\, a^\dagger(\mathbf{p})\, \mathrm{i}\mathbf{p} - \dots \\
&\quad \dots - a^\dagger(\mathbf{p})\, a(\mathbf{p})\, (-\mathrm{i}\mathbf{p}) - a^\dagger(\mathbf{p})\, a^\dagger(-\mathbf{p})\, (-\mathrm{i}\mathbf{p})\Big).
\end{aligned} \tag{2.118}$$

Using that $a(\mathbf{p})\, a(-\mathbf{p})\, \mathbf{p}$ is symmetric under $\mathbf{p} \to -\mathbf{p}$ we see, that the first and last term in the integral vanish, because $\int \mathrm{d}^3p\; a(\mathbf{p})\, a(-\mathbf{p})\, \mathbf{p} = 0$. With ordered operators we get

$$\begin{aligned}
\mathrm{P}^i &= \frac{1}{2} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \Big(2\, a^\dagger(\mathbf{p})\, a(\mathbf{p})\, bfp + \big[a^\dagger(\mathbf{p}), a(\mathbf{p})\big]\; \mathbf{p}\Big) \\
&= \frac{1}{2} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \Big(2\, a^\dagger(\mathbf{p})\, a(\mathbf{p})\, bfp - (2\pi)^3 \delta(\mathbf{0})\, \mathbf{p}\Big).
\end{aligned} \tag{2.119}$$

Analogously to Eq. (2.88) the second term formally diverges. We justify dropping it by arguing, that again only *differences* in momenta are important and no absolute momentum can be defined. Finally, we obtain the

4-momentum operator

$$\begin{aligned}
\mathrm{P}^0 &= H \\
\text{(spatial momentum)} \quad \mathbf{P} &= \int \mathrm{d}^3x\, \Pi\, \nabla\phi \\
&\simeq \int \frac{\mathrm{d}^3p}{(2\pi)^3} \mathbf{p}\, a^\dagger(\mathbf{p})\, a(\mathbf{p}) \quad \text{with} \quad \mathbf{P}\,|\mathbf{p}\rangle = \mathbf{p}\,|\mathbf{p}\rangle\,.
\end{aligned} \tag{2.120}$$

Lastly, we discuss **Lorentz symmetry** in the Fock space. Let $U(\Lambda)$ denote the unitary Fock space representation of a Lorentz transformation Λ. Then

$$\begin{aligned}
U(\Lambda)\,|0\rangle &= |0\rangle \\
U(\Lambda)\,|\mathbf{p}\rangle &= |\Lambda\mathbf{p}\rangle\,.
\end{aligned} \tag{2.121}$$

Note, that

$$\langle \mathbf{q}|\mathbf{p}\rangle = 2\omega_{\mathbf{p}}(2\pi)^3\, \delta(\mathbf{p} - \mathbf{q}) \tag{2.122}$$

is Lorentz invariant, as

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, 2\pi\, \delta(p^2 - m^2)\, \theta(p^0) \tag{2.123}$$

is invariant under proper orthochronous Lorentz transformations $\left(\det\Lambda = 1,\, \Lambda^0{}_0 > 0\right)$, and

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}}\, 2\omega_{\mathbf{p}} (2\pi)^3\, \delta(\mathbf{p}-\mathbf{q}) = 1\,. \tag{2.124}$$

With this, we have completed the Fock space construction. Recall, that $\phi(x)$ generates a superposition of one particle states from the vacuum (Eq. (2.107)). Further we remark, that causality is encoded in the propagator

$$D(x-y) = \langle 0|\phi(x)\,\phi(y)|0\rangle \tag{2.125}$$

and its variants. This is further discussed in chapter 3, section I. To finish, we consider the quantisation for **complex scalar fields**. We have

$$\begin{aligned}
S[\phi] &= \int \mathrm{d}^4 x \left(\partial_\mu \phi\, \partial^\mu \phi^* - m^2 \phi\, \phi^*\right) \quad \text{with} \quad \phi = \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\,\phi_2) \\
\phi(\mathbf{x}) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}\mathbf{px}} + b^\dagger(\mathbf{p})\, \mathrm{e}^{\mathrm{i}\mathbf{px}}\right) \\
\Pi(\mathbf{x}) &= \partial^0 \phi^*(x) = -\mathrm{i} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(b(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}\mathbf{px}} - a^\dagger(\mathbf{p})\, \mathrm{e}^{\mathrm{i}\mathbf{px}}\right)
\end{aligned} \tag{2.126}$$

with the commutation relations

$$\begin{aligned}
[\phi(\mathbf{x}), \Pi(\mathbf{y})] &= \delta(\mathbf{x}-\mathbf{y}) \\
\left[a(\mathbf{p}), a^\dagger(\mathbf{q})\right] &= (2\pi)^3\, \delta(\mathbf{p}-\mathbf{q}) \\
\left[b(\mathbf{p}), b^\dagger(\mathbf{q})\right] &= (2\pi)^3\, \delta(\mathbf{p}-\mathbf{q}) \\
&.
\end{aligned} \tag{2.127}$$

Note, that the other commutators vanish. The Hamiltonian

$$H = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\tilde{\Pi}(\mathbf{p})\, \tilde{\Pi}^*(\mathbf{p}) + \omega_{\mathbf{p}}^2\, \tilde{\phi}(\mathbf{p})\, \tilde{\phi}^*(\mathbf{p})\right) \tag{2.128}$$

becomes the operator

$$H = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(a^\dagger(\mathbf{p})\, a(\mathbf{p}) + b^\dagger(\mathbf{p})\, b(\mathbf{p})\right), \tag{2.129}$$

where the integrand corresponds to the sum of the energy of particles and antiparticles.
The Noether charge from Eq. (2.61) is then given by

$$
\begin{aligned}
Q &= i \int \mathrm{d}^3 x \left(\phi^* \partial_t \phi - (\partial_t \phi^*)\, \phi \right) \\
&= i \int \mathrm{d}^3 x \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}\, 2\omega_{\mathbf{q}}}} \cdot ... \\
& \quad ... \cdot \Big((a^\dagger(\mathbf{p})\, \mathrm{e}^{\mathbf{ipx}} + b(\mathbf{p})\, \mathrm{e}^{-\mathbf{ipx}}) \cdot (-\mathrm{i}\omega_{\mathbf{q}}\, a(\mathbf{q})\, \mathrm{e}^{-\mathbf{iqx}} + \mathrm{i}\omega_{\mathbf{q}}\, b^\dagger(\mathbf{q})\, \mathrm{e}^{\mathbf{iqx}}) - ... \\
& \quad ... - (\mathrm{i}\omega_{\mathbf{p}}\, a^\dagger(\mathbf{p})\, \mathrm{e}^{\mathbf{ipx}} - \mathrm{i}\omega_{\mathbf{p}}\, b(\mathbf{p})\, \mathrm{e}^{-\mathbf{ipx}}) \cdot (a(\mathbf{q})\, \mathrm{e}^{-\mathbf{iqx}} + b^\dagger(\mathbf{q})\, \mathrm{e}^{\mathbf{iqx}} \Big) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \Big(a^\dagger(\mathbf{p})\, a(\mathbf{p}) - b^\dagger(\mathbf{p})\, b(\mathbf{p}) \Big), \qquad (2.130)
\end{aligned}
$$

and

$$
\langle \alpha | Q | \alpha \rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \Big(\alpha^*\, \alpha(\mathbf{p}) - \beta^*\, \beta(\mathbf{p}) \Big). \qquad (2.131)
$$

3. Perturbation Theory

Perturbation theory is a standard method in quantum field theory. It considers **interaction as a perturbation of the free theory**. Thus, we assume $\lambda \ll 1$ and expand the observables, e.g. scattering amplitudes, in order of λ.

I. Interaction Picture

Introductory to this section, Heisenberg and Schrödinger picture are recapitulated, as they are commonly used in the 1+0 dimensional theory (QM). Subsequently the interaction picture is discussed, in which both, operators and states, evolve in time.
In the previous chapter, the Fock space construction was performed in the Heisenberg picture. In the **Heisenberg picture** the operators evolve in time, whereas the states are stationary:

$$\begin{aligned} \mathrm{i}\partial_t \left| f \right\rangle &= 0 \\ \mathrm{i}\partial_t \mathcal{O}(t) &= [\mathcal{O}(t), H] \,, \end{aligned} \tag{3.1}$$

with

$$\mathcal{O}(t) = \mathrm{e}^{\mathrm{i}Ht} \, \mathcal{O} \, \mathrm{e}^{-\mathrm{i}Ht} \,. \tag{3.2}$$

Indeed, the field operator $\phi(x)$ follows from

$$\begin{aligned} \phi(x) &= \mathrm{e}^{\mathrm{i}Ht} \, \phi(\mathbf{x}) \, \mathrm{e}^{-\mathrm{i}Ht} \\ &= \mathrm{e}^{\mathrm{i}Ht} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(a(\mathbf{p}) \, \mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}} + a^\dagger(\mathbf{p}) \, \mathrm{e}^{-\mathrm{i}\mathbf{p}\mathbf{x}} \Big) \mathrm{e}^{-\mathrm{i}Ht} \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(a(\mathbf{p}) \, \mathrm{e}^{-\mathrm{i}(\omega_{\mathbf{p}} t - \mathbf{p}\mathbf{x})} + a^\dagger(\mathbf{p}) \, \mathrm{e}^{\mathrm{i}(\omega_{\mathbf{p}} t - \mathbf{p}\mathbf{x})} \Big) \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(a(\mathbf{p}) \, \mathrm{e}^{-\mathrm{i}px} + a^\dagger(\mathbf{p}) \, \mathrm{e}^{\mathrm{i}px} \Big), \end{aligned} \tag{3.3}$$

with

$$\mathrm{e}^{\mathrm{i}Ht} \, a(\mathbf{p}) \, \mathrm{e}^{-\mathrm{i}Ht} = a(\mathbf{p}) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{p}} t} \,. \tag{3.4}$$

Eq. (3.4) follows from:

$$H \, a(\mathbf{p}) - a(\mathbf{p}) \, (H - \omega_{\mathbf{p}}) \,, \tag{3.5}$$

with

$$H = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \, a^\dagger(\mathbf{p}) \, a(\mathbf{p}) \,. \tag{3.6}$$

Now we use, that

$$\begin{aligned}
[H, a(\mathbf{p})] &= \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \omega_{\mathbf{q}} \left[a^\dagger(\mathbf{q})\, a(\mathbf{q}), a(\mathbf{p})\right] \\
&= \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \omega_{\mathbf{q}} \left(a^\dagger(\mathbf{q})\ [a(\mathbf{q}), a(\mathbf{p})] + \left[a^\dagger(\mathbf{q}), a(\mathbf{p})\right]\ a(\mathbf{q})\right) \\
\text{using Eq. (2.83)} \quad \rightarrow \quad &= -\omega_{\mathbf{p}}\, a(\mathbf{p})\,, \qquad (3.7)
\end{aligned}$$

to find

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}Ht}\, a(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}Ht} &= a(\mathbf{p})\, \mathrm{e}^{\mathrm{i}(H-\omega_{\mathbf{p}})t}\, \mathrm{e}^{-\mathrm{i}Ht} \\
&= a(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{p}} t}\,. \qquad (3.8)
\end{aligned}$$

Similarly one shows

$$\mathrm{e}^{\mathrm{i}Ht}\, a^\dagger(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}Ht} = a^\dagger(\mathbf{p})\, \mathrm{e}^{\mathrm{i}\omega_{\mathbf{p}} t}\,. \qquad (3.9)$$

A different approach is the **Schrödinger picture**, where the states evolve in time and the operators are stationary:

$$\begin{aligned}
\mathrm{i}\partial_t \left|f(t)\right\rangle &= H \left|f\right\rangle \\
\mathrm{i}\partial_t O &= 0\,, \qquad (3.10)
\end{aligned}$$

with

$$\left|f(t)\right\rangle = \mathrm{e}^{-\mathrm{i}Ht} \left|f\right\rangle\,. \qquad (3.11)$$

Hence, the time evolution operator $U(t, t') := \mathrm{e}^{-\mathrm{i}H(t-t')}$ either acts on the operators (Heisenberg) *or* the states (Schrödinger).
At this point we remark, that **causality is encoded in the operator** $\phi(x)$. To show this, we consider

$$\begin{aligned}
[\phi(x), \phi(y)] &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}\, 2\omega_{\mathbf{q}}}} \cdot ... \\
&\quad ... \cdot \left(\left[a(\mathbf{p}), a^\dagger(\mathbf{q})\right] \mathrm{e}^{-\mathrm{i}(p-q)x} + \left[a^\dagger(\mathbf{p}), a(\mathbf{q})\right] \mathrm{e}^{\mathrm{i}(p-q)x}\right) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \mathrm{e}^{-\mathrm{i}p(x-y)} - \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \mathrm{e}^{\mathrm{i}p(x-y)} \\
\text{using Eq. (2.21)} \quad \rightarrow \quad &= \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\, \delta(p^2 - m^2)\, \theta(p^0)\, \mathrm{e}^{-\mathrm{i}p(x-y)} - ... \\
&\quad ... - \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\, \delta(p^2 - m^2)\, \theta(p^0)\, \mathrm{e}^{\mathrm{i}p(x-y)} \qquad (3.12)
\end{aligned}$$

We now take into account, that both terms in Eq. (3.12) are Lorentz invariant measures. Further we use, that for space-like separation $((x-y)^2 < 0$ a Lorentz transformation with

$$\Lambda(x - y) = -(x - y) \qquad (3.13)$$

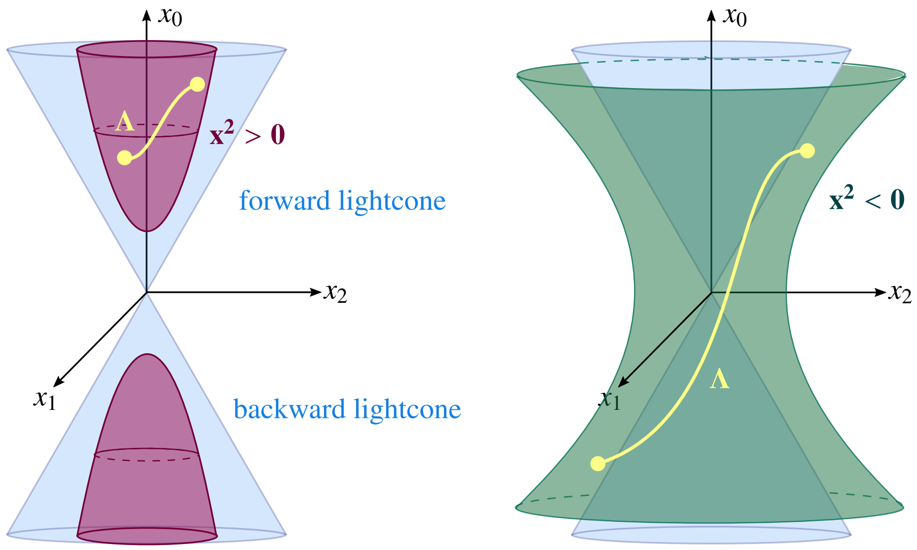


Figure 3.1.: A Lorentz transformation can connect arbitrary points on a x^2 = const. surface in the Minkowski diagram. Thus, a Lorentz transformation for space-like separations with $\Lambda(x-y) = -(x-y)$ exists, which is demonstrated on the right hand side of the sketch. The left hand side shows, that this is not possible for time-like separations.

exists (see figure 3.1). Hence,

$$\begin{aligned} & \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\, \delta(p^2 - m^2)\, \theta(p^0)\, \mathrm{e}^{\mathrm{i}p(x-y)} \\ = & \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\, \delta(p^2 - m^2)\, \theta(p^0)\, \mathrm{e}^{\mathrm{i}p\,\Lambda(x-y)} \\ = & \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\, \delta(p^2 - m^2)\, \theta(p^0)\, \mathrm{e}^{-\mathrm{i}p(x-y)} \quad \text{for} \quad (x-y)^2 < 0\,. \end{aligned} \tag{3.14}$$

Using this in Eq. (3.12) yields for a real scalar field

$$[\phi(x), \phi(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0\,. \tag{3.15}$$

This implies, that **the order of observations or measurements with a space-like relationship do not impact each other**. Analogously, for complex scalar field we find

$$\left[\phi(x), \phi^\dagger(y)\right] = 0 \quad \text{for} \quad (x-y)^2 < 0\,. \tag{3.16}$$

In the **interaction picture** now both, operators and states, evolve in time. We decompose the Lagrangian density in a free and an interaction part

$$\begin{aligned} \mathcal{L}(\phi) &= \mathcal{L}_0(\phi) + \mathcal{L}_{\text{int}}(\phi) \\ &= \frac{1}{2}\phi(x)\left(-\partial^2 - m^2\right)\phi(x) + \mathcal{L}_{\text{int}}(\phi)\,, \end{aligned} \tag{3.17}$$

where

$$\mathcal{L}_{\text{int}}(\phi) = -V(\phi) \tag{3.18}$$

is a polynomial in ϕ. The same follows for the Hamiltonian density

$$\begin{aligned} \mathcal{H}(\Pi,\phi) &= \mathcal{H}_0(\Pi,\phi) + \mathcal{H}_{\text{int}}(\phi) \\ &= \frac{1}{2}\Pi(x)^2 + \frac{1}{2}\phi(x)\left(-\Delta + m^2\right)\phi(x) + \mathcal{H}_{\text{int}}(\phi)\,, \end{aligned} \tag{3.19}$$

where

$$\mathcal{H}_{\text{int}}(\phi) = V(\phi)\,. \tag{3.20}$$

Commonly, one defines the

interaction part of the Hamiltonian density

$$H_{\text{int}}(\phi) = V(\phi) = \frac{\lambda}{4!}\,\phi(x)^4\,. \tag{3.21}$$

Note, that the normalisation factor of 4! can differ in literature. This definition can be justified, by considering, that the quadratic term of ϕ is already included in the free field. The next higher polynomial, ϕ^3, would spoil the symmetry $\phi \mapsto -\phi$. Also higher terms than ϕ^4 can be excluded in 1+3 dimensional theory, due to renormalisability. Thus, ϕ^4-theory is the "working horse" of quantum field theory. In the interaction picture, the operators evolve in time with the free Hamiltonian

$$\begin{aligned} \mathrm{i}\partial_t \mathcal{O} &= [\mathcal{O}, H_0] \\ \Rightarrow \mathcal{O}(t) &= \mathrm{e}^{\mathrm{i}H_0 t}\,\mathcal{O}\,\mathrm{e}^{-\mathrm{i}H_0 t}\,, \end{aligned} \tag{3.22}$$

with

$$H_0 = \int \mathrm{d}^3x\,\mathcal{H}_0\,. \tag{3.23}$$

On the other hand, the states evolve with the interaction Hamiltonian

$$\mathrm{i}\partial_t \left|f\right\rangle = H_{\text{int}} \left|f\right\rangle\,. \tag{3.24}$$

Note, that

$$\begin{aligned} [H_0, H_{\text{int}}] &\neq 0 \\ \Rightarrow \partial_t H_{\text{int}} &\neq 0 \quad \text{i.e.} \quad H_{\text{int}} = H_{\text{int}}(t)\,. \end{aligned} \tag{3.25}$$

Time evolution of a state can also be expressed as

$$\left|f(t)\right\rangle = U(t,t_0)\left|f(t_0)\right\rangle\,, \tag{3.26}$$

where $U(t,t_0)$ is the unitary time-evolution operator. With Eq. (3.22) we find the

time evolution of U(t, t₀)

$$\mathrm{i}\partial_t U(t,t_0) = H_{\text{int}}(t)\,U(t,t_0)\,. \tag{3.27}$$

We remark, that the scattering matrix is defined as

S-matrix

$$S = \lim_{\substack{t_0 \to -\infty \\ t \to +\infty}} U(t,t_0)\,. \tag{3.28}$$

Strictly speaking, this means λ is adiabatically switched on and off. Thus, initial state $\left|i\right\rangle$ and final state

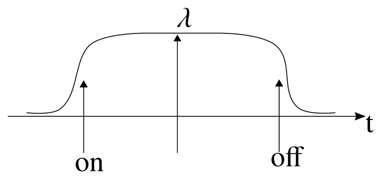


Figure 3.2.: Sketch of adiabatic switch on/off of λ, i.e. interaction.

$|f\rangle$ are given by:

$$
\begin{aligned}
|\text{state } t \to -\infty\rangle &= |i\rangle \\
|\text{state } t \to +\infty\rangle &= |f\rangle \,.
\end{aligned} \tag{3.29}
$$

Note, that for a proper treatment of the S-matrix the LSZ-formalism is used.
Next, we derive the explicit expression for $U(t, t_0)$. For this purpose, we take the infinitesimal form of Eq. (3.24) and use it to rewrite the state $|f(t)\rangle$ iteratively:

$$
\begin{aligned}
|f(t+\Delta t)\rangle &= |f(t)\rangle - \mathrm{i}\,\Delta t\, H_{\text{int}}(t)\;|f(t)\rangle \\
&= \Big(1 - \mathrm{i}\,\Delta t\, H_{\text{int}}(t)\Big)\;|f(t)\rangle \\
&= \Big(1 - \mathrm{i}\,\Delta t\, H_{\text{int}}(t)\Big)\Big(1 - \mathrm{i}\,\Delta t\, H_{\text{int}}(t-\Delta t)\Big)\;|f(t-\Delta t)\rangle \\
&\;\vdots \\
&= \prod_{n=0}^{N}\Big(1 - \mathrm{i}\,\Delta t\, H_{\text{int}}(t - n\,\Delta t)\Big)\;|f(t - N\,\Delta t)\rangle \,.
\end{aligned} \tag{3.30}
$$

Thus,

$$
U(t+\Delta t, t - N\Delta t) = \prod_{n=0}^{N}\Big(1 - \mathrm{i}\,\Delta t\, H_{\text{int}}(t - n\,\Delta t)\Big). \tag{3.31}
$$

We expand in powers of Δt:

$$
\begin{aligned}
U(t+\Delta t, t - N\Delta t) &= 1 + (-\mathrm{i})\Delta t \sum_{n=0}^{N} H_{\text{int}}(t - n\,\Delta t) + \ldots \\
&\qquad \ldots + (-\mathrm{i})^2 (\Delta t)^2 \sum_{n<m} H_{\text{int}}(t - n\,\Delta t)\, H_{\text{int}}(t - m\,\Delta t) + \ldots \,.
\end{aligned} \tag{3.32}
$$

Note, that $n < m$ in the second sum corresponds to the time 'on the left' being larger than the time 'on the right' (time ordering). Now let $\Delta t \to 0$ with $N\,\Delta t = t - t_0$. Then Eq. (3.32) becomes

$$
\begin{aligned}
&1 + (-\mathrm{i}) \int_{t_0}^{t} \mathrm{d}t'\; H_{\text{int}}(t') + \ldots \\
&\qquad \ldots + (-\mathrm{i})^2 \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t''\; H_{\text{int}}(t')\, H_{\text{int}}(t'') + \ldots \,,
\end{aligned} \tag{3.33}
$$

where the first integral in the last line corresponds to the sum over n and the second integral to the sum over m. The integral limits give an equivalent ordering to $n < m$.
Finally, we obtain the

time-evolution operator

$$U(t,t_0) = T \exp\left(-\mathrm{i}\int_{t_0}^{t} \mathrm{d}t'\ H_{\mathrm{int}}(t')\right) \quad \text{for} \quad t > t_0\,, \tag{3.34}$$

with the time ordering operator

$$T\ A(t)\,B(t') = A(t)\,B(t')\,\theta(t-t') + B(t')\,A(t)\,\theta(t'-t)\,. \tag{3.35}$$

Example 4: Time ordering for the second order term of $\mathbf{U(t, t_0)}$.

This example shows, how the time ordering operator acts on the second order term in the expansion of Eq. (3.34), yielding the second order term of Eq. (3.33).

$$\begin{aligned} &\frac{1}{2} T \int_{t_0}^{t} \mathrm{d}t'\ H_{\mathrm{int}}(t') \int_{t_0}^{t} \mathrm{d}t''\ H_{\mathrm{int}}(t'') \\ =\ &\frac{1}{2}\left(\int_{t_0}^{t} \mathrm{d}t'\ H_{\mathrm{int}}(t') \int_{t_0}^{t'} \mathrm{d}t''\ H_{\mathrm{int}}(t'') + \dots \quad (t' < t)\right. \\ &\left.\dots + \int_{t_0}^{t} \mathrm{d}t''\ H_{\mathrm{int}}(t'') \int_{t_0}^{t''} \mathrm{d}t'\ H_{\mathrm{int}}(t')\right) \quad (t'' > t') \\ =\ &\int_{t_0}^{t} \mathrm{d}t'\ H_{\mathrm{int}}(t') \int_{t_0}^{t'} \mathrm{d}t''\ H_{\mathrm{int}}(t'')\,. \end{aligned} \tag{3.36}$$

This works analogously for higher order terms, where $n!$ equal terms cancel with the $\frac{1}{n!}$ factor from the expansion.

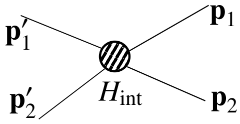
Note, that

$$H_{\mathrm{int}} = \int \mathrm{d}^3x\,\phi^4(x) \sim a^2\,(a^\dagger)^2\,. \tag{3.37}$$

Hence, the interaction Hamiltonian creates two particles and annihilates them, leading to infinite vacuum processes $\langle 0|H_{\mathrm{int}}|0\rangle$.

Example 5: 2 to 2 scattering.

$\langle \mathbf{p_1'}\, \mathbf{p_2'} | \sim$ at $t \to +\infty$



$\sim |\mathbf{p_1}\, \mathbf{p_2}\rangle$ at $t \to -\infty$

The S-matrix is given by

$$S = \mathbb{1} + \mathrm{i}\, T\,, \tag{3.38}$$

where the unity matrix represents the part without scattering. For 2 to 2 scattering it is

$$\mathrm{i}\, T_{fi} \cong -\mathrm{i}\, \langle 0 | a(\mathbf{p_1'})\, a(\mathbf{p_2'}) \frac{\lambda}{4!} \int \mathrm{d}^4 x\, \phi^4(x)\, a^\dagger(\mathbf{p_1})\, a^\dagger(\mathbf{p_2}) | 0 \rangle\,, \tag{3.39}$$

where we have already dropped the infinite constants and used the first order interaction term, only. To explicitly compute $\mathrm{i}\, T_{fi}$, we use Eq. (2.83) and perform normal ordering (pull all creation operators in H_{int} to the left and all annihilation operators to the right). Using Eq. (2.90) all operators then vanish and only the terms with the commutators remain. Then we obtain

$$\mathrm{i}\, T_{fi} =: \mathrm{i}\, M (2\pi)^4\, \delta(p_1 + p_2 - p_1' - p_2') \quad \text{with} \quad \mathrm{i}\, M = \mathrm{i}\, \lambda\,, \tag{3.40}$$

where M denotes the matrix element.
In the following, normal ordered expressions will be marked by colons, e.g.

$$:a(\mathbf{p_1})\, a^\dagger(\mathbf{p_2}): \,=\, a^\dagger(\mathbf{p_2})\, a(\mathbf{p_1})\,. \tag{3.41}$$

For instance, normal ordering discards the infinite vacuum terms of the free Hamiltonian H_0 in Eq. (2.87), as

$$\begin{aligned} :H_0: \;&=\; \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} : \frac{1}{2}\, a^\dagger(\mathbf{p})\, a(\mathbf{p}) + \frac{1}{2}\, a(\mathbf{p})\, a^\dagger(\mathbf{p}) : \\ &=\; \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} a^\dagger(\mathbf{p})\, a(\mathbf{p})\,. \end{aligned} \tag{3.42}$$

Further, the normal ordered interaction Hamiltonian H_{int} already yields Eq. (3.40):

$$\begin{aligned} &\frac{\lambda}{4!} \int \mathrm{d}^4 x \;: \phi(x)^4 : \\ &\sim \frac{\lambda}{4!} : (a^\dagger)^2\, a^2 + a^\dagger\, a\, a^\dagger\, a + a\, a^\dagger\, a\, a^\dagger + \ldots \\ &\ldots + a^\dagger\, a^2\, a^\dagger + a\, (a^\dagger)^2\, a + a^2\, (a^\dagger)^2 : \\ &\sim \frac{\lambda}{4} (a^\dagger)^2\, a^2\,. \end{aligned} \tag{3.43}$$

Then,

$$
\begin{aligned}
& \frac{\lambda}{4}\left(\prod_i \int \frac{\mathrm{d}^3 q_i}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{q_i}}}\right) \mathrm{e}^{-\mathrm{i}x(q_3+q_4-q_1-q_2)} \sqrt{2\omega_{\mathbf{p_1'}}\, 2\omega_{\mathbf{p_2'}}\, 2\omega_{\mathbf{p_1}}\, 2\omega_{\mathbf{p_2}}} \cdot \ldots \\
& \ldots \cdot \langle 0| a(\mathbf{p_1'})\, a(\mathbf{p_2'}) \left(a^\dagger(\mathbf{q_1})\, a^\dagger(\mathbf{q_2})\, a(\mathbf{q_3})\, a(\mathbf{q_4})\right) a^\dagger(\mathbf{p_1})\, a^\dagger(\mathbf{p_2}) |0\rangle \\
= \; & 4 \cdot \frac{\lambda}{4}\left(\prod_i \int \frac{\mathrm{d}^3 q_i}{(2\pi)^3}\sqrt{\frac{2\omega_{q_i}}{2\omega_{q_i}}}\right) \mathrm{e}^{-\mathrm{i}x(q_3+q_4-q_1-q_2)} \cdot \ldots \\
& \ldots \cdot \delta(\mathbf{p_1'} - \mathbf{q_1})\, \delta(\mathbf{p_2'} - \mathbf{q_2})\, \delta(\mathbf{p_1} - \mathbf{q_3})\, \delta(\mathbf{p_2} - \mathbf{q_4}) \\
= \; & \lambda\, \mathrm{e}^{-\mathrm{i}x(p_1+p_2-p_1'-p_2')} \,,
\end{aligned} \tag{3.44}
$$

with e.g.

$$
\begin{aligned}
a(\mathbf{q_4})\, a^\dagger(\mathbf{p_1})\, a^\dagger(\mathbf{p_2})\, |0\rangle &= \left[a(\mathbf{q_4}), a^\dagger(\mathbf{p_1})\right] a^\dagger(\mathbf{p_2})\, |0\rangle + a^\dagger(\mathbf{p_1})\, a(\mathbf{q_4})\, a^\dagger(\mathbf{p_2})\, |0\rangle \\
&= \left((2\pi)^3\, \delta(\mathbf{q_4} - \mathbf{p_1})\, a^\dagger(\mathbf{p_2}) + a^\dagger(\mathbf{p_1}) \left[a(\mathbf{q_4}), a^\dagger(\mathbf{p_2})\right] + a^\dagger(\mathbf{p_1})\, a^\dagger(\mathbf{p_2})\, a(\mathbf{q_4})\right) |0\rangle \\
&= (2\pi)^3 \left(\delta(\mathbf{q_4} - \mathbf{p_1})\, a^\dagger(\mathbf{p_2}) + \delta(\mathbf{q_4} - \mathbf{p_2})\, a^\dagger(\mathbf{p_1})\right).
\end{aligned} \tag{3.45}
$$

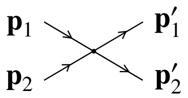
Lastly, using

$$
\int \mathrm{d}^4 x\, \mathrm{e}^{-\mathrm{i}x(p_1+p_2-p_1'-p_2')} = \delta(p_1 + p_2 - p_1' - p_2')\,, \tag{3.46}
$$

we obtain Eq. (3.40).
The difference between interaction Hamiltonian and normal ordered interaction Hamiltonian consequently gives the vacuum contributions:

$$
H_{\text{int}} = \,:H_{\text{int}}:\, + \frac{\lambda}{8}\int \mathrm{d}^4 x \left(\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\right)^2 + (a^\dagger a, a\, a^\dagger)\text{-terms}\,. \tag{3.47}
$$

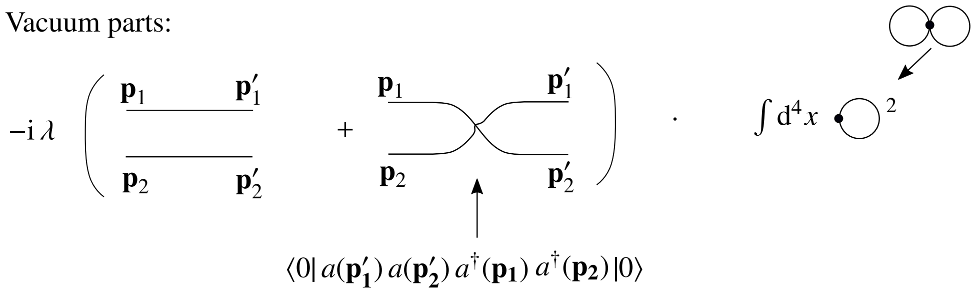
Let us now consider the interpretation of these terms:

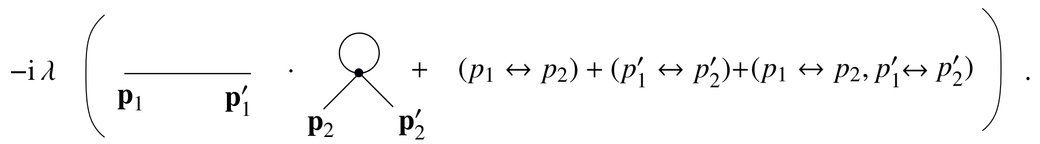


$: \quad -\mathrm{i}\,\lambda \quad \cdot \quad (2\pi)^4\, \delta(p_1 + p_2 - p_1' - p_2')$

interaction strength   4-momentum conservation

Vacuum parts:





Note, that for the first term

$$
\begin{aligned}
&\langle 0|a(\mathbf{p}_1')\, a(\mathbf{p}_2')\, a^\dagger(\mathbf{p}_1)\, a^\dagger(\mathbf{p}_2)|0\rangle \left(1 - \mathrm{i}\,\lambda \int \mathrm{d}^4x\, \mathrm{O}^2\right) \\
= \;& \langle 0|a(\mathbf{p}_1')\, a(\mathbf{p}_2')\, a^\dagger(\mathbf{p}_1)\, a^\dagger(\mathbf{p}_2)|0\rangle \left(\exp\left(-\mathrm{i}\,\lambda \int \mathrm{d}^4x\, \mathrm{O}^2\right) + O(\lambda^2)\right) . \qquad (3.48)
\end{aligned}
$$

It can be shown, that the second order term is an infinite *phase*, that contains all vacuum processes. Nevertheless, as the phase/loops are infinite, the call for an appropriate treatment. Commonly one uses regularisation and renormalisation ("theory in a box", see chapter 7).

Next, we discuss a core ingredient of perturbation theory: the **propagator**.
Starting in the Heisenberg picture, we introduce the vacuum of the full theory $|\Omega\rangle$, with

$$
\mathrm{i}\partial_t\, |\Omega\rangle = 0\,. \qquad (3.49)
$$

In the Heisenberg picture the operators evolve with the full Hamiltonian, i.e.

$$
\mathrm{i}\partial_t \phi_H = [\phi_H, H]\;, \qquad (3.50)
$$

with

$$
\phi_H = \mathrm{e}^{-\mathrm{i}Ht}\, \phi(0,\mathbf{x})\, \mathrm{e}^{\mathrm{i}Ht}\,. \qquad (3.51)
$$

We now link this to the interaction picture, where the states evolve with H_{int} and

$$
\begin{aligned}
|f(t)\rangle_I &= U(t,0)\,|f(0)\rangle_I \\
\mathrm{i}\partial_t \phi_I &= [\phi_I, H_0]\;. \qquad (3.52)
\end{aligned}
$$

Hence,

$$
\phi_I(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left(a(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}px} + a^\dagger(\mathbf{p})\, \mathrm{e}^{\mathrm{i}px}\right)_{p_0=\omega_\mathbf{p}}\,. \qquad (3.53)
$$

Using, that

$$
U(t,0) = \mathrm{e}^{\mathrm{i}H_0 t}\, \mathrm{e}^{-\mathrm{i}Ht}\,, \qquad (3.54)
$$

it follows

$$
\phi_H(x) = U(0,x^0)\, \phi_I(x)\, U(x^0,0)\,, \qquad (3.55)
$$

with

$$
\begin{aligned}
\phi_H(x)\,|f\rangle_H &= U(0,x^0)\, \phi_I(x)\, U(x^0,0)\,|f\rangle_H \\
\mathrm{i}\partial_t U(x^0,0)\,|f\rangle_H &= H_{\text{int}}\, U(x^0,0)\,|f\rangle_H\;. \qquad (3.56)
\end{aligned}
$$

It is tempting to identify $U(x^0,0)\,|f\rangle_H$ with the interaction picture states $|f(t)\rangle_I$. At $t \to \pm\infty$, λ is switched off adiabatically, and $|f\rangle_I$ tend to free in/out states. Considering, that $U(0,\infty) = U(\infty,0)^{-1}$, we have

$$
\begin{aligned}
\langle\Omega|\, U(0,x^0) &= \langle\Omega|\, U(0,\infty)\, U(\infty,x^0) \\
&= \sum_n \langle\Omega|\, U(0,\infty)\, |n\rangle_I\ {}_I\langle U|\,(\infty,x^0) \\
&= \langle\Omega|\, U(0,\infty)\, |0\rangle\, \langle 0|\, U(\infty,x^0)\,, \qquad (3.57)
\end{aligned}
$$

where in the last step we used, that *adiabatically* indicates: $\left|\text{n-particles}\right\rangle_{\text{free}} \xrightarrow{U} \left|\text{n-particles}\right\rangle_{\text{full}}$. Also, it is

$$U(x^0,0)\,|\Omega\rangle = U(x^0,-\infty)\,|0\rangle\,\langle 0|\,U(-\infty,0)\,|\Omega\rangle\ . \tag{3.58}$$

Further note, that

$$\begin{aligned} \mathrm{i}\partial_t U(t,0) &= H_I(t)\,U(t,0)\,, \\ H_I(t) &= \mathrm{e}^{\mathrm{i}H_0 t}\,H_{\text{int}}\,\mathrm{e}^{-\mathrm{i}H_0 t} \\ &= \frac{\lambda}{4!}\int \mathrm{d}^3x\,\phi_I(x)^4\,, \\ \mathrm{i}\partial_t H_I &= [H_I, H_0]\ . \end{aligned} \tag{3.59}$$

Thus,

$$\begin{aligned} \mathrm{i}\partial_t\phi_H(x) &= U(0,t)\,H_{\text{int}}(x^0)\,\phi_I(t)\,U(t,0) - U(0,t)\,\phi_I\,H_{\text{int}}(x^0)\,U(t,0) + ... \\ &\quad ... + U(0,t)\,\left[\phi_I(x), H_{\text{int}}(t)\right]\,U(t,0) \\ &= 0\,. \end{aligned} \tag{3.60}$$

We now compute the propagator

$$\begin{aligned} &\langle\Omega|T\ \phi_H(x)\phi_H(y)|\Omega\rangle \\ =\ &\langle\Omega|\phi_H(x)\phi_H(y)|\Omega\rangle\ \theta(x^0-y^0) + \langle\Omega|\phi_H(y)\phi_H(x)|\Omega\rangle\ \theta(y^0-x^0)\,. \end{aligned} \tag{3.61}$$

For $x^0 > 0 > y^0$:

$$\begin{aligned} && &\langle\Omega|T\ \phi_H(x)\phi_H(y)|\Omega\rangle \\ \text{using Eq. (3.55)} &\quad\rightarrow & =\ &\langle\Omega|U(0,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,U(y^0,0)|\Omega\rangle \\ && =\ &\langle 0|U(\infty,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,U(y^0,-\infty)|0\rangle\cdot ... \\ && &...\cdot\frac{1}{\Big(\langle 0|\,U(0,\infty)\,|\Omega\rangle\cdot\langle 0|\,U(-\infty,0)\,|\Omega\rangle\Big)^{-1}}\,, \end{aligned} \tag{3.62}$$

where we used, that in general

$$U(x^0,z^0) = U(x^0,y^0)\,U(y^0,z^0) \quad \text{for} \quad x^0 > y^0 > z^0\,. \tag{3.63}$$

This follows straightforwardly from Eq. (3.34). Note, that the dominator in Eq. (3.62) is (a product of two) phases, i.e. $|\,\langle\Omega|\,U(0,\infty)\,|0\rangle\,| = 1$. This becomes evident, when considering:

$$|\,\langle\Omega|\,U(0,x^0)| = 1\,, \tag{3.64}$$

as from U being unitary it follows

$$|\,\langle\Omega|\,U(0,x^0)|^2 = \langle\Omega|\,U(0,x^0)\,U^\dagger(0,x^0)\,|\Omega\rangle = \langle\Omega|\Omega\rangle = 1\,. \tag{3.65}$$

And analogously

$$|\langle 0|\, U(-\infty, x^0)|^2 = 1\,. \tag{3.66}$$

Combining Eq. (3.65) and Eq. (3.66) yields

$$|\langle \Omega|\, U(0,\infty)\,|0\rangle\,| = 1\,. \tag{3.67}$$

Hence,

$$\langle \Omega|\, U(0,\infty)\,|0\rangle^{-1} = \langle \Omega|\, U(0,\infty)\,|0\rangle^{*} = \langle \Omega|\, U^{\dagger}(0,\infty)\,|0\rangle\ , \tag{3.68}$$

i.e. the normalisation factor in Eq. (3.62) is a phase. Likewise to Eq. (3.57), we use the adiabaticity and get

$$\begin{aligned}
& \langle \Omega|\, U(0,\infty)\,|0\rangle^{-1}\,\langle 0|\, U(-\infty,0)\,|\Omega\rangle^{-1} \\
\text{using Eq. (3.68)} \quad \rightarrow \quad =\ & \langle 0|\, U(\infty,0)\,|\Omega\rangle\,\langle \Omega|\, U(0,-\infty)\,|0\rangle \\
=\ & \langle 0|U(\infty,0)\, U(0,-\infty)|0\rangle = \langle 0|U(-\infty,\infty|0\rangle \\
=\ & \langle 0|S\,|0\rangle = \langle 0|T\,\exp\left(-\mathrm{i}\int \mathrm{d}t\, H_{\text{int}}(t)\right)|0\rangle\ .
\end{aligned} \tag{3.69}$$

We also have for the numerator of Eq. (3.62)

$$\begin{aligned}
& \langle 0|U(\infty,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,U(y^0,-\infty)|0\rangle \\
=\ & \langle 0|T\ \phi_I(x)\,\phi_I(y)\exp\left(-\mathrm{i}\int \mathrm{d}t\, H_{\text{int}}(t)\right)|0\rangle\ .
\end{aligned} \tag{3.70}$$

Finally, with $\phi_I = \phi$, and the analogous result for $y^0 > x^0$, we obtain for the

propagator (two-point function)

$$\langle \Omega|T\ \phi_H(x)\,\phi_H(y)|\Omega\rangle = \frac{\langle 0|T\ \phi_I(x)\,\phi_I(y)\exp\left(-\mathrm{i}\int \mathrm{d}t\, H_{\text{int}}(t)\right)|0\rangle}{\langle 0|T\ \exp\left(-\mathrm{i}\int \mathrm{d}t\, H_{\text{int}}(t)\right)|0\rangle}\ . \tag{3.71}$$

This is straightforwardly extended to the

propagator (n-point function)

$$\langle \Omega|T\ \phi_H(x_1)\cdots\phi_H(x_n)|\Omega\rangle = \frac{\langle 0|T\ \phi_H(x_1)\cdots\phi_H(x_n)\exp\left(-\mathrm{i}\int \mathrm{d}t\, H_{\text{int}}(t)\right)|0\rangle}{\langle 0|T\ \exp\left(-\mathrm{i}\int \mathrm{d}t\, H_{\text{int}}(t)\right)|0\rangle}\ . \tag{3.72}$$

Note, that the denominators in Eq. (3.71) and Eq. (3.72) are phases. For example, the linear term in λ is

$$-\mathrm{i}\,\langle 0|\int \mathrm{d}t\, H_{\text{int}}|0\rangle = -\mathrm{i}\,\lambda\,\langle 0|\int \mathrm{d}^4x\,\phi(x)^4|0\rangle = -\frac{\mathrm{i}}{8}\int \mathrm{d}^4x\left(\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{q}}}\right)^2 , \tag{3.73}$$

which cancels the vacuum term in Eq. (3.47). We remark, that both, phase factor (denominator) and the vacuum contributions in the nominator, are infinite and cancel.

II. Wick's Theorem

We have seen, that the computation of scattering amplitudes relates to the computation of time ordered n-point functions

$$\langle 0|T\ \phi(x_1)\cdots\phi(x_n)\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4y\ \mathcal{L}_{\mathrm{int}}(y)}|0\rangle\ , \tag{3.74}$$

where

$$-\int \mathrm{d}t\ H_{\mathrm{int}} = \int \mathrm{d}t\ L_{\mathrm{int}} = \int \mathrm{d}^4y\ \mathcal{L}_{\mathrm{int}}(y)\,, \tag{3.75}$$

and the coupling $\lambda \ll 1$. Since

$$\langle 0|T\ \phi(x_1)\cdots\phi(x_n)\prod_{i=1}^{m}\mathcal{L}_{\mathrm{int}}(y_i)|0\rangle = \frac{1}{(4!)^m}\langle 0|\phi(x_1)\cdots\phi(x_n)\,\phi(x_{n+1})\cdots\phi(x_{n+4m})|0\rangle\ , \tag{3.76}$$

with $x_{n+1},\ldots,x_{n+4} = y_1;\cdots;x_{n+4(m-1)},\ldots,x_{n+4m} = y_m$, the only building block in Eq. (3.74) is

$$\boxed{\langle 0|T\ \phi(x_1)\cdots\phi(x_n)|0\rangle\ .} \tag{3.77}$$

For $x_1^0 > \cdots > x_n^0$ Eq. (3.77) reduces to $\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle$, and we simply have to use the canonical commutation relations (Eq. (2.83)) (note, that $\phi = \phi_I$ is free). In the next step, we use normal ordering and the vanishing expectation value of the ordered parts. For the two-point function this works as follows: Firstly, we rewrite

$$\phi(x) = \phi_+(x) + \phi_-(x)\,, \tag{3.78}$$

with

$$\begin{aligned}\phi_+(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\,a^\dagger(\mathbf{p})\,\mathrm{e}^{\mathrm{i}px}\\ \phi_-(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\,a(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}px}\,.\end{aligned} \tag{3.79}$$

For $x^0 > y^0$ it is

$$\begin{aligned}T\ \phi(x)\,\phi(y) &= \phi_+(x)\,\phi_+(y) + \phi_+(x)\,\phi_-(y) + \phi_-(x)\,\phi_+(y) + \phi_-(x)\,\phi_-(y)\\ &= \phi_+(x)\,\phi_+(y) + \phi_+(x)\,\phi_-(y) + \Big(\phi_+(y)\,\phi_-(x) + [\phi_-(x),\phi_+(y)]\Big) + \phi_-(x)\,\phi_-(y)\,.\end{aligned} \tag{3.80}$$

Thus,

$$T\ \phi(x)\,\phi(y)\Big|_{x_0>y_0} =\ :\phi(x)\,\phi(y):\ + [\phi_-(x),\phi_+(y)]\ , \tag{3.81}$$

where

$$:\phi_-(x)\,\phi_+(y):\ = \phi_+(y)\,\phi_-(x) \quad \forall x\,, \tag{3.82}$$

from

$$:a(\mathbf{p})\,a^\dagger(\mathbf{q}):\ = a^\dagger(\mathbf{q})\,a(\mathbf{p})\,. \tag{3.83}$$

Then, by taking the vacuum expectation values, the normal ordered part vanishes. The time ordered propagator for the two-point function is called

Feynman-propagator

$$\begin{aligned}\mathcal{D}_F(x-y) &= \langle 0|T\,\phi(x)\phi(y)|0\rangle \\ &= [\phi_-(x),\phi_+(y)]\,\theta(x^0-y^0)+[\phi_-(y),\phi_+(x)]\,\theta(y^0-x^0)\,,\end{aligned} \tag{3.84}$$

and is the key-ingredient in (time ordered) perturbation theory. To explicitly calculate the Fenyman-propagator we consider

$$\begin{aligned}&[\phi_-(x),\phi_+(y)]\,\theta(x^0-y^0)\\ =\;& \int\frac{\mathrm{d}^3q}{(2\pi)^3}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}\,2\omega_{\mathbf{q}}}}\left[a(\mathbf{p}),a^\dagger(\mathbf{q})\right]\mathrm{e}^{-\mathrm{i}\,(px+qy)}\,\theta(x^0-y^0)\\ &\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\mathrm{e}^{-\mathrm{i}\,p(x-y)}\,\theta(x^0-y^0)\\ \Rightarrow\;& \mathcal{D}_F(x-y)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\Big(\mathrm{e}^{-\mathrm{i}\,p(x-y)}\,\theta(x^0-y^0)+\mathrm{e}^{\mathrm{i}\,p(x-y)}\,\theta(y^0-x^0)\Big).\end{aligned} \tag{3.85}$$

Using

$$\theta(x)\cong\lim_{\epsilon\to0}\int\mathrm{d}p\,\mathrm{e}^{\mathrm{i}\,pt}\frac{1}{p+\mathrm{i}\,\epsilon}\,, \tag{3.86}$$

the Feynman propagator can be written as

Feynman-propagator (explicit)

$$\mathcal{D}_F(x-y) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\,\epsilon}\mathrm{e}^{-\mathrm{i}\,p(x-y)}\,,\quad \text{as}\quad \epsilon\to0\,. \tag{3.87}$$

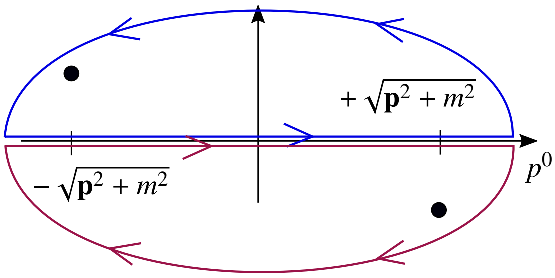
This can be proven with the **Residue theorem**:
The contour integral of a function $f(z)$ around a closed, counterclockwise path encircling a domain where $f(z)$ has a finite number of isolated singularities (poles at $z=z_i$, $i=1,2,\ldots,n$) is

$$\oint\mathrm{d}z\,f(z)=2\pi\,\mathrm{i}\sum_{i=1}^{n}\mathrm{Res}(f,z_i)\,, \tag{3.88}$$

where the residue of $f(z)$ at a simple pole z_i is $\mathrm{Res}(f,z_i)=\lim_{z\to z_i}(z-z_i)\,f(z)$.

The integrand in Eq. (3.87) has poles at $(p^0)^2=\pm\sqrt{\mathbf{p}^2+m^2-\mathrm{i}\,\epsilon}$, as shown in figure 3.3. Hence, for $x^0-y^0>0$ the pole is at $p^0_-=\sqrt{\mathbf{p}^2+m^2-\mathrm{i}\,\epsilon}\to\omega_{\mathbf{p}}$, and thus

$$\begin{aligned}\mathcal{D}_F(x-y) &= -\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{2\pi\,\mathrm{i}}{2\pi}\mathrm{res}_{p^0_-}\left(\frac{\mathrm{e}^{-\mathrm{i}\,p(x-y)}}{p^2-m^2+\mathrm{i}\,\epsilon}\right)\\ &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{i}\,\frac{\mathrm{e}^{-\mathrm{i}\,p(x-y)}}{2\mathrm{i}\,\omega_{\mathbf{p}}}\bigg|_{p^0=\omega_{\mathbf{p}}}\,.\end{aligned} \tag{3.89}$$



$x^0 - y^0 < 0$: close contour in upper half plane

$x^0 - y^0 > 0$: close contour in lower half plane

Figure 3.3.: Sketch of the poles of the integrand of Eq. (3.87) and the contour for the Residue theorem.

This works similarly for $x^0 - y^0 < 0$, and proves Eq. (3.87).
Note, that we have parametrised the time ordered propagator in terms of commutators. On operator level we have

$$\begin{aligned} T\,\phi(x)\,\phi(y) &= \;:\phi(x)\,\phi(y): \;+ \ldots \\ &\quad \ldots + [\phi_-(x), \phi_+(y)]\;\Theta(x^0 - y^0) + [\phi_-(y), \phi_+(x)]\;\Theta(y^0 - x^0)\,, \end{aligned} \tag{3.90}$$

i.e.

$$T\,\phi(x)\,\phi(y) = \;:\phi(x)\,\phi(y):\; + \overline{\phi(x)\phi}(y)\,, \tag{3.91}$$

with the *contraction*

$$\begin{aligned} \overline{\phi(x)\,\phi}(y) &= [\phi_-(x), \phi_+(y)]\;\Theta(x^0 - y^0) + [\phi_-(y), \phi_+(x)]\;\Theta(y^0 - x^0) \\ &= \mathcal{D}_F(x-y)\,. \end{aligned} \tag{3.92}$$

Note, that $\mathcal{D}_F(x-y)$ is a c-number (and not an operator!). We use this, to generalise the time ordering to a product of n fields. This is

Wick's theorem

$$T\,\phi(x_1)\cdots\phi(x_n) = \;:\phi(x_1)\cdots\phi(x_n) + \text{all contractions}:\,, \tag{3.93}$$

where

$$\begin{aligned} &\phi(x_1)\cdots\overline{\phi(x_i)\cdots\phi}(x_j)\cdots\phi(x_n) \\ &= \phi(x_1)\cdots\phi(x_{i-1})\,\phi(x_{i+1})\cdots\phi(x_{j-1})\,\phi(x_{j+i})\cdots\phi(x_n)\,\overline{\phi(x_i)\,\phi}(x_j) \end{aligned} \tag{3.94}$$

Example 6: 4-point correlation function.

$$
\begin{aligned}
T\,\phi(x_1)\cdots\phi(x_4) &= T\,\phi_1\,\phi_2\,\phi_3\,\phi_4 \\
&= \;:\phi_1\,\phi_2\,\phi_3\,\phi_4 + \wick{\c1\phi_1\,\c1\phi_2}\,\phi_3\,\phi_4 + \phi_1\,\phi_2\,\wick{\c1\phi_3\,\c1\phi_4} + \wick{\c1\phi_1\,\phi_2\,\c1\phi_3}\,\phi_4 + ... \\
&\quad ... + \wick{\c1\phi_1\,\phi_2\,\phi_3\,\c1\phi_4} + \phi_1\,\wick{\c1\phi_2\,\c1\phi_3}\,\phi_4 + \phi_1\,\wick{\c1\phi_2\,\phi_3\,\c1\phi_4} + ... \\
&\quad ... + \wick{\c1\phi_1\,\c1\phi_2\,\c2\phi_3\,\c2\phi_4} + \wick{\c1\phi_1\,\c2\phi_2\,\c1\phi_3\,\c2\phi_4} + \wick{\c1\phi_1\,\c2\phi_2\,\c2\phi_3\,\c1\phi_4}\;:\,,
\end{aligned}
\tag{3.95}
$$

where e.g.

$$
:\wick{\c1\phi_1\,\c1\phi_2}\,\phi_3\,\phi_4: \;=\; :\phi_3\,\phi_4:\;\wick{\c1\phi_1\,\c1\phi_2} \;=\; :\phi_3\,\phi_4:\;\mathcal{D}_F(x_1-x_2)\,. \tag{3.96}
$$

Note also that

$$
\langle 0|:\mathcal{O}:|0\rangle = 0\,. \tag{3.97}
$$

With this it follows

$$
\begin{aligned}
&\langle 0|T\,\phi(x_1)\cdots\phi(x_4)|0\rangle \\
&= \mathcal{D}_F(x_1-x_2)\,\mathcal{D}_F(x_2-x_3) + ... \\
&\quad ... + \mathcal{D}_F(x_1-x_3)\,\mathcal{D}_F(x_2-x_4) + ... \\
&\quad ... + \mathcal{D}_F(x_1-x_4)\,\mathcal{D}_F(x_2-x_3)\,,
\end{aligned}
\tag{3.98}
$$

where each term corresponds to one of the terms with two contractions in Eq. (3.95).

It remains to prove Wick's theorem. We will do this by induction. First we show, that it holds for the one- and two-point function:

$$
\begin{aligned}
n = 1,2: \qquad T\,\phi_1 &= \;:\phi_1: \\
T\,\phi_1\,\phi_2 &= \;:\phi_1\,\phi_2: + \wick{\c1\phi_1\,\c1\phi_2}\,.
\end{aligned}
\tag{3.99}
$$

Next, we assume that Wick's theorem applies to the n-point function, i.e. $T\,\phi_2\cdots\phi_{n+1}$. Without loss of generality we can assume that $x_1^0 \geq x_i^0\ \forall i$. Then

$$
\begin{aligned}
T\,\phi_1\cdots\phi_{n+1} &= \phi_1 T\,\phi_2\cdots\phi_{n+1} \\
&= \phi_1\Big(:\phi_2\cdots\phi_{n+1} + \text{all contractions}:\Big) \\
&= \Big(\phi_{1_+} + \phi_{1_-}\Big)\Big(:\phi_2\cdots\phi_{n+1} + \text{all contractions}:\Big) \\
&= \;:\phi_1\cdots\phi_{n+1} + \left[\phi_{1_-},\phi_2\right]\,\phi_3\cdots\phi_{n+1} + ... \\
&\quad ... + \phi_2\,\left[\phi_{1_-},\phi_3\right]\,\phi_4\cdots\phi_{n+1} + \cdots + \phi_2\cdots\left[\phi_{1_-},\phi_{n+1}\right]: \; + ... \\
&\quad ... + \Big(\phi_{1_+} + \phi_{1_-}\Big)\Big(:\text{all contractions}:\Big).
\end{aligned}
\tag{3.100}
$$

Using

$$[\phi_{1_-}, \phi_i] = [\phi_{1_-}, \phi_{i_+}] = \wick{\c{\phi}_1 \c{\phi}_i}\,, \tag{3.101}$$

and similarly as in Eq. (3.100) for

$$\Big(\phi_{1_+} + \phi_{1_-}\Big)\Big(\text{ : all contractions : }\Big), \tag{3.102}$$

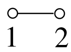
we obtain

$$T\ \phi(x_1)\cdots\phi(x_{n+1}) = :\phi(x_1)\cdots\phi(x_{n+1}) + \text{all contractions}:, \tag{3.103}$$

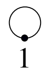
which completes the induction.

III. Feynman Rules

With Wick's theorem (Eq. (3.93)) we write every time ordered n-point function as product of Feynman propagators (Eq. (3.84)) plus the normal ordered terms. We introduce the diagrammatical notation

$$\mathcal{D}_F(x_1 - x_2) = \langle 0|T\ \phi_1\,\phi_2|0\rangle =$$  .

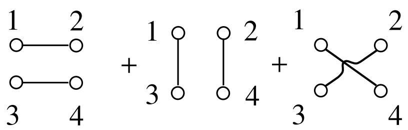
First of all, let us discuss the expression $\langle 0|T\ \phi_1\,\phi_1|0\rangle$. With Eq. (3.87), we find

$$\mathcal{D}_F(0) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\ \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\,\epsilon} =$$  .

This is a singularity, which will again be removed by an appropriate adjustment of the computation (renormalisation). In particular, we note that the momentum dimension of $\mathcal{D}_F(0)$ is two. Therefore, we argue

$$\mathcal{D}_F(0) = M^2 + \text{infinite}\,. \tag{3.104}$$

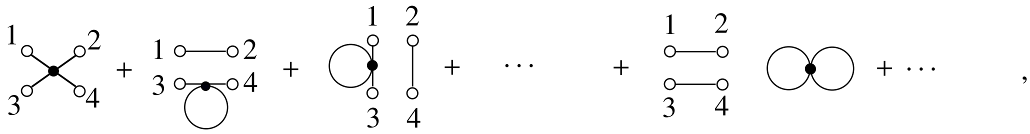
Now, let us again consider the 2-2 scattering, as it is a relevant example.
The zeroth order term in λ, i.e. the term without interaction is simply given by the expectation value of the 4-point function:

$\mathcal{O}(\lambda^0)$: $\qquad \langle 0|T\ \phi_1\,\phi_2\,\phi_3\,\phi_4|0\rangle =$  .

The first order term is

$$\begin{aligned}
&\frac{-\mathrm{i}\lambda}{4!}\int \mathrm{d}^4x\ \langle 0|T\ \phi_1\,\phi_2\,\phi_3\,\phi_4\ \phi\,\phi\,\phi\,\phi|0\rangle \\
= \ &\frac{-\mathrm{i}\lambda}{4!}\int \mathrm{d}^4x \Big[\wick{\c1{\phi}_1 \c1{\phi} \c2{\phi}_2 \c2{\phi} \c3{\phi}_3 \c3{\phi} \c4{\phi}_4 \c4{\phi}}\ \cdot 4! + \ldots \\
&\ldots + \wick{\c1{\phi}\c1{\phi}}\Big(\wick{\c2{\phi}_1 \c2{\phi} \c3{\phi}_2 \c3{\phi} \c4{\phi}_3 \c4{\phi}_4}\Big)\cdot(12\ \text{perm.}) + \ldots \\
&\ldots + \wick{\c1{\phi}\c1{\phi}\c2{\phi}\c2{\phi}}\Big(\wick{\c3{\phi}_1 \c3{\phi}_2 \c4{\phi}_3 \c4{\phi}_4}\Big)\cdot(3\ \text{perm.})\Big]\,.
\end{aligned} \tag{3.105}$$

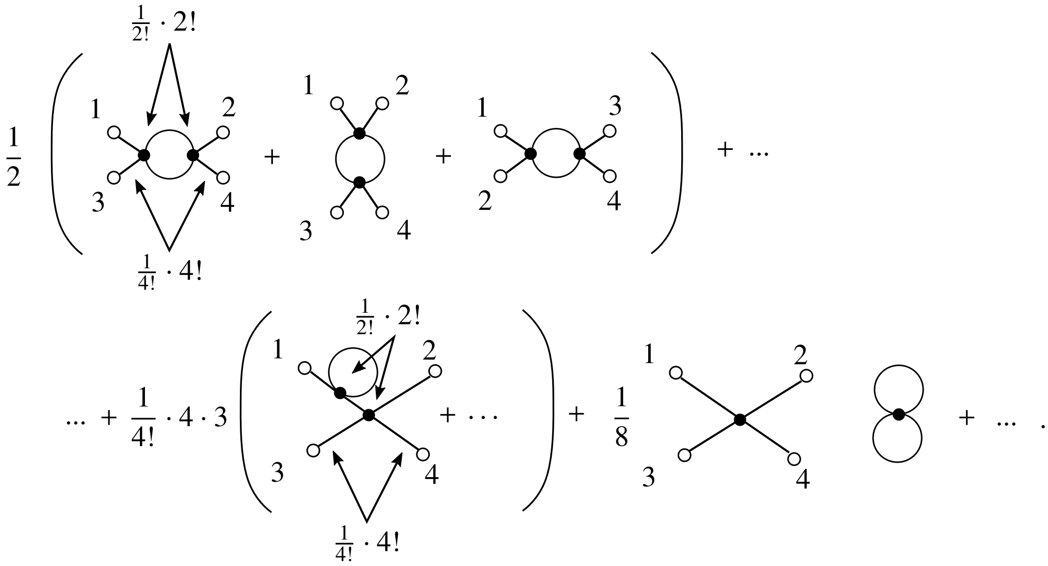
Note, that the factor 4! accounts for all possibilities to contract ϕ^4 with $\phi_1\cdots\phi_4$, and the factors 12 and 3 account for permutations of the contractions, that give an identical expression. This will again be further discussed below. Diagrammatically and without the symmetry factors this writes

$O(\lambda^1)$: 

where the vertices correspond to $\left(-\mathrm{i}\,\lambda \int \mathrm{d}^4x\right)$. The second order term

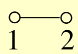
$$\left(\frac{-\mathrm{i}\lambda}{4!}\right)^2 \int \mathrm{d}^4x \int \mathrm{d}^4z \;\langle 0|T\;\phi_1\,\phi_2\,\phi_3\,\phi_4\;\phi(x)^4\;\phi(z)^4|0\rangle$$

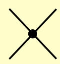
comprises

$O(\lambda^2)$: 

To encounter the right prefactor for each diagram, we need to do some combinatorics: The permutations of how to contract $\phi\,\phi\,\phi\,\phi$ in H_{int} with the external fields gives a factor 4!, which cancels with the denominator in $\frac{-\mathrm{i}\lambda}{4!}$. This originally motivated the normalisation in Eq. (3.21). When loops are present, we further have to account for the symmetries that result from contracting the ϕ^4 amongst each others in H_{int}. For this purpose we introduce the **symmetry factor** $\mathbf{\frac{1}{S}}$, where S corresponds to the number of interchanging components without changing the diagram.
Now we can write down the

Feynman rules (position space)

i)  $= \mathcal{D}_F(x_1 - x_2)$

ii)  $= (-\mathrm{i}\,\lambda)\int \mathrm{d}^4x$

iii) multiplication with $\dfrac{1}{S}$.

(3.106)

We will use this to obtain the final result for the vacuum expectation value (Eq. (3.72))

$$\langle T\;\phi_1\cdots\phi_n\rangle := \frac{\langle 0|T\;\phi_1\cdots\phi_n\exp\left(\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\right)|0\rangle}{\langle 0|T\;\exp\left(\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\right)|0\rangle}\,. \tag{3.107}$$

For the computation we note, that each term $\langle 0|T\ \phi_1 \cdots \phi_n\ \frac{(\mathcal{L}_{\text{int}})^m}{m!}|0\rangle$ can be ordered in terms of contractions between the ϕ_i and the $\mathcal{L}_{\text{int}}$'s:

$$
\begin{aligned}
\langle 0|T\ \phi_1 \cdots \phi_n\ \frac{(\mathcal{L}_{\text{int}})^m}{m!}|0\rangle \quad &= \quad \langle 0|T\ \phi_1 \cdots \phi_n|0\rangle\ \frac{1}{m!}\langle 0|T\ (\mathcal{L}_{\text{int}})^m|0\rangle + ... \\
&\quad ... + \langle 0|T\ \phi_1 \underbrace{\cdots \phi_n\ \mathcal{L}_{\text{int}}}|0\rangle\ \frac{1}{(m-1)!}\langle 0|T\ (\mathcal{L}_{\text{int}})^{m-1}|0\rangle + ...\,,
\end{aligned}
\tag{3.108}
$$

where "$\underbrace{\qquad}$" denotes all contractions, where internal fields from the interaction Hamiltonian are connected to *external* fields (and not amongst themselves).
We use that

$$
\begin{aligned}
&\frac{1}{m!}\langle 0|T\ \phi_1 \cdots \phi_n\,(\mathcal{L}_{\text{int}})^m|0\rangle\Big|_{O(2\mathcal{L}_{\text{int}}-contr.)} \\
=\ &\frac{1}{m!}\langle 0|T\ \phi_1 \underbrace{\cdots \phi_n\,(\mathcal{L}_{\text{int}}})^2|0\rangle\ \langle 0|T\ (\mathcal{L}_{\text{int}})^{m-2}|0\rangle \cdot \frac{m\cdot(m-1)}{2} \\
=\ &\frac{1}{2}\langle 0|T\ \phi_1 \underbrace{\cdots \phi_n\,(\mathcal{L}_{\text{int}}})^2|0\rangle\ \frac{1}{(m-2)!}\ \langle 0|T\ (\mathcal{L}_{\text{int}})^{m-2}|0\rangle\ ,
\end{aligned}
\tag{3.109}
$$

and that in general the combinatorics factor for $l-\mathcal{L}_{\text{int}}$-contractions is

$$
\frac{1}{m!}\binom{m}{l} = \frac{1}{m!}\frac{m!}{(m-l)!\,l!} = \frac{1}{(m-l)!\,l!}\,. \tag{3.110}
$$

Then,

$$
\begin{aligned}
\langle 0|T\ \phi_1 \cdots \phi_n \exp\left(\mathrm{i}\int \mathrm{d}^4x\ \mathcal{L}_{\text{int}}\right)|0\rangle \quad &= \quad \Big(\langle 0|T\ \phi_1 \cdots \phi_n|0\rangle + \langle 0|T\ \phi_1 \underbrace{\cdots \phi_n\ \mathcal{L}_{\text{int}}}|0\rangle + ... \\
&\quad ... + \langle 0|T\ \phi_1 \underbrace{\cdots \phi_n\,(\mathcal{L}_{\text{int}}})^2/2|0\rangle + ...\Big)\cdot ... \\
&\quad ... \cdot \langle 0|T\ \exp\left(\mathrm{i}\int \mathrm{d}^4x\ \mathcal{L}_{\text{int}}\right)|0\rangle\ .
\end{aligned}
\tag{3.111}
$$

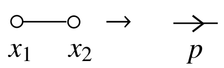
Consequently the denominator cancels all the vacuum terms. Therefore, the vacuum expectation value is given by

$$
\frac{\langle 0|T\ \phi_1 \cdots \phi_n \exp\left(\mathrm{i}\int \mathrm{d}^4x\ \mathcal{L}_{\text{int}}\right)|0\rangle}{\langle 0|T\ \exp\left(\mathrm{i}\int \mathrm{d}^4x\ \mathcal{L}_{\text{int}}\right)|0\rangle} = \langle 0|T\ \phi_1 \underbrace{\cdots \phi_n\ \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\ \mathcal{L}_{\text{int}}}}|0\rangle\ , \tag{3.112}
$$

which corresponds to all diagrams without "vacuum bubbles".
As most computations are carried out in **momentum space**, we will terminate this section, by examining the Fourier transforms.
The Feynman propagator becomes

$$
\mathcal{D}_F(x_1-x_2) \to \mathcal{D}_F(p) = \frac{\mathrm{i}}{p_1 - m^2 + \mathrm{i}\epsilon}(2\pi)^4\,\delta^4(p_1-p_2)\,, \tag{3.113}
$$

and



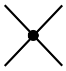
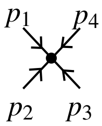
.

For the vertices we write

$$-\mathrm{i}\lambda \int \mathrm{d}^4x\, \phi(x)^4 = -\mathrm{i}\lambda \int \prod_{i=1}^{4} \frac{\mathrm{d}^4 p_i}{(2\pi)^4}\phi(p_i) \cdot (2\pi)^4\, \delta^4(p_1 + p_2 + p_3 + p_4)\,, \tag{3.114}$$

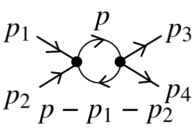
where the delta-function indicates momentum conservation. Hence,

$$-\mathrm{i}\lambda \quad \rightarrow \quad -\mathrm{i}\lambda\,(2\pi)^4\, \delta^4(p_1 + p_2 + p_3 + p_4)$$

 $\rightarrow$ 

$p_4 = -(p_1 + p_2 + p_3)$.

For example:

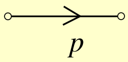


$$= (2\pi)^4\, \delta^4(p_1 + p_2 - p_3 - p_4) \cdot \ldots$$

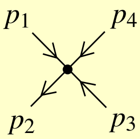
$$\ldots \cdot \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2 + \mathrm{i}\epsilon} \frac{1}{(p - p_1 - p_2)^2 - m^2 + \mathrm{i}\epsilon} \quad .$$

We now have the

Feynman rules (momentum space)

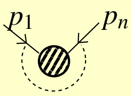
i) 

$= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$

ii) 

$= -\mathrm{i}\lambda$ and $p_4 = -(p_1 + p_2 + p_3)$ (momentum conservation)

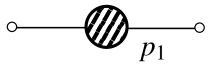
iii) $\displaystyle\int \frac{\mathrm{d}^4}{(2\pi)^4}$ for each loop

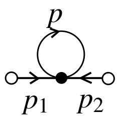
iv) $(2\pi)^4\, \delta^4(\sum_i p_i)$ for 

v) multiplication with $\dfrac{1}{S}$.

(3.115)

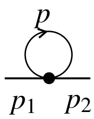
Example 7: two-point function in momentum space.

$$\langle T\,\phi(p_1)\,\phi(-p_2)\rangle = (2\pi)^4\,\delta^4(p_1+p_2)\;\cdot$$ 

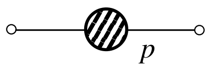
$$= \frac{\mathrm{i}}{p_1^2 - m^2 + \mathrm{i}\epsilon}\,(2\pi)^4\,\delta^4(p_1+p_2) + \frac{1}{2}$$  $$+\,\mathcal{O}(\lambda^2)\,.$$

$\frac{1}{S}$

Without external propagators it is:

 $$= -\mathrm{i}\lambda \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} = -\mathrm{i}\,\Pi + \mathcal{O}(\lambda^2)\,.$$

Heuristics:

 $$= \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} + \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}\,(-\mathrm{i}\,\Pi)\,\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} + \mathcal{O}(\lambda^2)$$

$$= \frac{\mathrm{i}}{p^2 - m^2 - \Pi + \mathrm{i}\epsilon} + \mathcal{O}(\lambda^2)\quad .$$

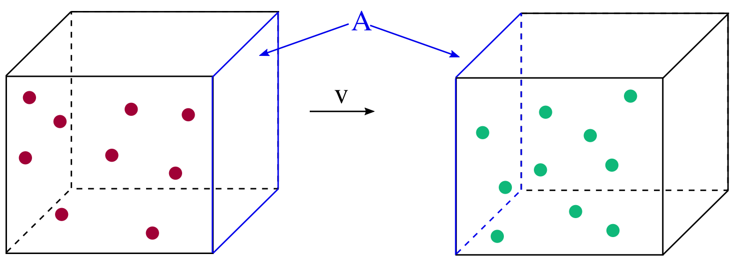
It follows, that we have an **interacting mass** $m^2 - \Pi$, which is finite. In general (beyond 1-loop) it holds:

$$\Pi \to \Pi(p) \tag{3.116}$$

The proper treatment is again provided through renormalisation and the LSZ-formalism.

IV. Cross Section

We start by considering an exemplary fixed target experiment:



bunch of particles, type B
bunch length l_B
density ρ_B
velocity v

bunch of particles, type A
bunch length l_A
density ρ_A
velocity 0

The cross section is defined as

$$\sigma = \frac{N_{\text{events}}}{(N_B \cdot N_A)/A}\,, \tag{3.117}$$

where A is the scattering area (transverse). With a space-dependent density

$$(N_B \cdot N_A)/A = \int\limits_A d^2x\, \rho_A(x)\, \rho_B(x)\, l_A\, l_B\,, \tag{3.118}$$

the cross section is

$$\sigma = \frac{N_{\text{events}}}{l_A\, l_B \int_A d^2x\, \rho_A(x)\, \rho_B(x)}\,, \tag{3.119}$$

or for constant densities

$$\frac{N_{\text{events}}}{l_A\, \rho_A \cdot l_B\, \rho_B \cdot A}\,. \tag{3.120}$$

For the above example, we need to consider states, that are localised in space/momentum. Therefore, we consider the wave packet from Eq. (2.94):

$$\left|f_{\mathbf{p}}\right\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \mathrm{f}_{\mathbf{p}}(\mathbf{k})\, |\mathbf{k}\rangle\ , \tag{3.121}$$

with $\mathrm{f}_{\mathbf{p}}(\mathbf{k})$ being a packet at $\mathbf{p}$, e.g.

$$\mathrm{f}_{\mathbf{p}}(\mathbf{k}) \sim \mathrm{e}^{-(\mathbf{k}-\mathbf{p})^2/N} \quad \text{Gaussian}\,. \tag{3.122}$$

Using the normalisation, it follows

$$\begin{aligned} 1 \;&=\; \left\langle f_{\mathbf{p}}\middle|f_{\mathbf{p}}\right\rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \frac{1}{2\omega_{\mathbf{k}'}}\, \mathrm{f}^*{}_{\mathbf{p}}(\mathbf{k}')\, \mathrm{f}_{\mathbf{p}}(\mathbf{k})\, \left\langle \mathbf{k}'\middle|\mathbf{k}\right\rangle \\ &=\; \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} |\mathrm{f}_{\mathbf{p}}(\mathbf{k})|^2 \quad \text{(see Eq. (2.95))}\,. \end{aligned} \tag{3.123}$$

The Gaussian is localised in $\mathbf{k}$ and $\mathbf{x}$. Recall, that a Fourier transform of a Gaussian remains a Gaussian. In operator language,

$$\left|f_{\mathbf{p}}\right\rangle = \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{1}{2}\, \mathrm{e}^{i\mathbf{kx}}\, \mathrm{f}(\mathbf{k})\, \phi(\mathbf{x})\, |0\rangle - \mathrm{i} \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}}\, \mathrm{e}^{i\mathbf{kx}}\, \mathrm{f}(\mathbf{k})\, \Pi(\mathbf{x})\, |0\rangle\ , \tag{3.124}$$

where we used

$$|\mathbf{k}\rangle = \sqrt{2\omega_{\mathbf{k}}}\, a^\dagger\, |0\rangle \tag{3.125}$$

and Eq. (2.82), i.e.

$$a^\dagger(\mathbf{k}) = \int d^3x\, \mathrm{e}^{\mathrm{i}\mathbf{kx}} \Big(\omega_{\mathbf{k}}\, \phi(\mathbf{x}) - \mathrm{i}\, \Pi(\mathbf{x})\Big). \tag{3.126}$$

This indicates, that the probability for one particle is

$$\int \frac{d^3p}{(2\pi)^3} \left|\left\langle \mathbf{q}\middle|f_{\mathbf{p}}\right\rangle\right|^2 = \int \frac{d^3p}{(2\pi)^3} |f_{\mathbf{p}}(\mathbf{q})|^2 = 1\,. \tag{3.127}$$

In our case, the initial state is given by

$$|i\rangle = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k_A}}\, 2\omega_{\mathbf{k_B}}}\, \mathrm{f}_{\mathbf{p_A}}(\mathbf{k_A})\, \mathrm{f}_{\mathbf{p_B}}(\mathbf{k_B})\, |\mathbf{k_A}\, \mathbf{k_B}\rangle\ , \tag{3.128}$$

with

$$|\mathbf{k_A}\, \mathbf{k_B}\rangle = \sqrt{2\omega_{\mathbf{k_A}}\, 2\omega_{\mathbf{k_B}}}\, a^\dagger(\mathbf{k_A})\, a^\dagger(\mathbf{k_B})\, |0\rangle\ . \tag{3.129}$$

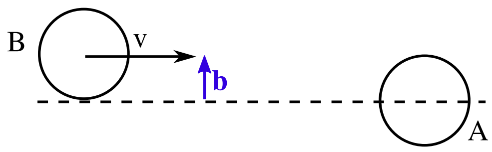


Figure 3.4.: Sketch of the impact parameter **b**.

Next, we introduce the impact parameter **b** (see figure 3.4). For this purpose, we recall, that the momentum operator **P** from Eq. (2.51) generates translations. Thus,

$$\begin{aligned} e^{-i\,\mathbf{P}\,\mathbf{b}}\,|\mathbf{k}\rangle &= e^{-i\,\mathbf{k}\mathbf{b}}\,|\mathbf{k}\rangle \\ \Rightarrow e^{-i\,\mathbf{P}\,\mathbf{b}}\,\big|f_{\mathbf{p}}\big\rangle &= \int d^3x \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}}\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{b})}\cdot\Big(\omega_{\mathbf{k}}\,\phi(\mathbf{x}) - i\,\Pi(\mathbf{x})\Big)|0\rangle\,. \end{aligned} \tag{3.130}$$

With this we rewrite the

initial state (with impact parameter **b**)

$$|i_{\mathbf{b}}\rangle = \int \frac{d^3k_A}{(2\pi)^3}\int\frac{d^3k_B}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k_A}}\,2\omega_{\mathbf{k_B}}}\,f_{\mathbf{p_A}}(\mathbf{k_A})\,f_{\mathbf{p_B}}(\mathbf{k_B})\,e^{-i\mathbf{k_B}\mathbf{b}}\,|\mathbf{k_A}\,\mathbf{k_B}\rangle\,. \tag{3.131}$$

This shows, that the impact parameter only gives an additional phase shift. The transition amplitude is given by

$$\langle \mathbf{p_{f_1}}\,\mathbf{p_{f_2}}\big|\,S\,|i_{\mathbf{b}}\rangle\,, \tag{3.132}$$

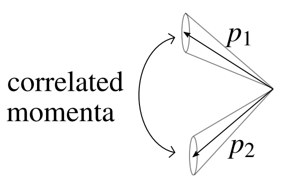
with probability $|\langle \mathbf{p_{f_1}}\,\mathbf{p_{f_2}}\big|\,S\,|i_{\mathbf{b}}\rangle|^2$. In the following we will restrict ourselves to a collision of the bunch with a *single target*, i.e. $N_A = 1$. Then integration over the impact area A is equal to integration over the impact parameter **b** and the number of events in a dense beam is

$$N_{\text{events}} = \frac{N_B}{A}\int_A d^2b\,|\langle \mathbf{p_{f_1}}\,\mathbf{p_{f_2}}\big|\,S\,|i_{\mathbf{b}}\rangle|^2\,. \tag{3.133}$$

With Eq. (3.117) it follows

$$\sigma(\mathbf{p_{f_1}},\mathbf{p_{f_2}}) = \int_A d^2b\,|\langle \mathbf{p_{f_1}}\,\mathbf{p_{f_2}}\big|\,S\,|i_{\mathbf{b}}\rangle|^2\,. \tag{3.134}$$

However, more realistic is a detection of a momentum region v_f, as detectors will in practice never be aligned with the beam. Hence, we will rather obtain something like:



It follows

$$\sigma(v_f) = \int\frac{d^3p_{f_1}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p_{f_1}}}}\int\frac{d^3p_{f_2}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p_{f_2}}}}\int d^2b\,|\langle \mathbf{p_{f_1}}\,\mathbf{p_{f_2}}\big|\,S\,|i_{\mathbf{b}}\rangle|^2\,, \tag{3.135}$$

where

$$\int\frac{d^3p_{f_1}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p_{f_1}}}} \sim \int\frac{d^4p_{f_1}}{(2\pi)^4}\,(2\pi)\,\delta(p_{f_1}^2 - m^2) \tag{3.136}$$

implies that it is on-shell, i.e. that $p^2 = m^2$. Finally, we obtain the

differential cross section (for n particles)

$$\mathrm{d}\sigma = \prod_{i=1}^{n} \frac{\mathrm{d}^3 p_{f_i}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p_{f_i}}}} \int \mathrm{d}^2 b \, |\langle \mathbf{p_{f_1}} \cdots \mathbf{p_{f_n}} | S \, |i_\mathbf{b}\rangle \, |^2 \, . \tag{3.137}$$

We assume now, that the $\mathbf{p_i}$ are not parallel to $\mathbf{p_B}$, so that there is no (trivial) forward scattering. Using

$$S_{fi} = \mathbb{1}_{fi} + \mathrm{i}\, T_{fi} \, , \quad \mathrm{i}\, T_{fi} = \mathrm{i}\, M_{fi}\, (2\pi)^4 \, \delta^4 \left(\sum p_{fi} - \sum k_i \right) , \tag{3.138}$$

we conclude

$$\begin{aligned} \mathrm{d}\sigma \;\; = \;\; & \prod_i \frac{\mathrm{d}^3 p_{fi}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p_i}}} \int \mathrm{d}^2 b \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k_A}}} \frac{\mathrm{d}^3 k_B}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k_B}}} \cdot \ldots \\ & \ldots \cdot \int \frac{\mathrm{d}^3 k'_A}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k'_A}}} \frac{\mathrm{d}^3 k'_B}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k'_B}}} \, \mathrm{f}_{\mathbf{p_A}}(\mathbf{k_A}) \, \mathrm{f}_{\mathbf{p_B}}(\mathbf{k_B}) \, \mathrm{f}^*_{\mathbf{p_A}}(\mathbf{k'_A}) \, \mathrm{f}^*{}_{\mathbf{p_B}}(\mathbf{k'_B}) \cdot \ldots \\ & \ldots \cdot \mathrm{e}^{-\mathrm{i}\mathbf{b}(\mathbf{k'_B} - \mathbf{k_B})} \, |M_{fi}|^2 \, (2\pi)^4 \, \delta^4 \left(\sum p_{fi} - \sum k_i \right) \cdot (2\pi)^4 \, \delta^4 \left(\sum p_{fi} - \sum k'_i \right) , \end{aligned} \tag{3.139}$$

with $k_1 = k_A$, $k_2 = k_B$.
To explicitly compute this, we first consider the integral over the impact parameter, as solely the phase factor depends on $\mathbf{b}$. Therefore,

$$\int \mathrm{d}^2 b \; \mathrm{e}^{\mathrm{i}\mathbf{b}(\mathbf{k'_B} - \mathbf{k_B})} = (2\pi)^2 \, \delta^2(\mathbf{k'_{B_\perp}} - \mathbf{k_{B_\perp}}) \, . \tag{3.140}$$

Next we examine the integral over the primed momenta,

$$\begin{aligned} & \int \mathrm{d}^3 k'_A \mathrm{d}^3 k'_B \, \delta^4 \left(\sum p_{fi} - \sum k'_i \right) \delta^2(\mathbf{k'_{B_\perp}} - \mathbf{k_{B_\perp}}) \\ = \;\; & \int \mathrm{d}\big(k_A^3\big)' \mathrm{d}\big(k_B^3\big)' \, \delta\Big(\sum p^3_{fi} - \sum \big(k_i^3\big)' \Big) \, \delta\Big(\sum p^0_{fi} - \sum \big(k_i^0\big)' \Big) \\ = \;\; & \int \mathrm{d}\, \big(k_A^3\big)' \, \delta\Big(\sum p^0_{fi} - \sum \big(k_i^0\big)' \Big) \, , \end{aligned} \tag{3.141}$$

with $k'_{B_\perp} = k_{B_\perp}$, $k'_{A_\perp} = k_{A_\perp}$, $\big(k_B^3\big)' = \sum p^3_{fi} - \big(k_A^3\big)'$. It follows

$$\begin{aligned} & \int \mathrm{d}^3 k'_A \mathrm{d}^3 k'_B \, \delta^4 \left(\sum p_{fi} - \sum k'_i \right) \delta^2(\mathbf{k'_{B_\perp}} - \mathbf{k_{B_\perp}}) \\ = \;\; & \int \mathrm{d}\, \big(k_A^3\big)' \, \delta\Big(\sum p^0_{fi} - \sqrt{(\mathbf{k_A}')^2 + m_A^2} - \sqrt{(\mathbf{k_B}')^2 + \mathbf{m_B^2}} \Big) \Big|_{\substack{k_{A/B_\perp} = k'_{A/B_\perp} \\ \sum k_i^3 = \sum (k_i^3)' \\ \sum k_i^0 = \sum (k_i^0)'}} \\ = \;\; & \frac{1}{\left| \frac{(k_A^3)'}{(k_A^0)'} - \frac{(k_B^3)'}{(k_B^0)'} \right|} \quad \overset{k_{A/B} = p_{A/B}}{\rightarrow} \quad \frac{1}{|v_A - v_B|} \, . \end{aligned} \tag{3.142}$$

where we also used, that $(\mathbf{k'_A})^2 = k_{A_\perp}^2 + \left(\left(k_A^3\right)'\right)^2$ and $(\mathbf{k'_B})^2 = k_{B_\perp}^2 + \left(\sum p_i^3 - \left(k_A^3\right)'\right)^2$.
As the wave packages $f_{\mathbf{p_{A/B}}}$ are located around $\mathbf{p_{A/B}}$, we can substitute $k'_{A/B} \to p_{A/B}$ in all prefactors. Then we obtain

$$\mathrm{d}\sigma = \prod_i \frac{\mathrm{d}^3 p_{fi}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p_{fi}}}} \frac{1}{4p_A^0\, p_B^0\, |v_A - v_B|} \cdot \ldots$$

$$\ldots \cdot \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3} \int \frac{\mathrm{d}^3 k_B}{(2\pi)^3} \frac{1}{2k_A^0\, 2k_B^0} \left|f_{\mathbf{p_A}}(\mathbf{k_A})\right|^2 \left|f_{\mathbf{p_B}}(\mathbf{k_B})\right|^2 \cdot \ldots \quad (= 1)$$

$$\ldots \cdot \left|M_{fi}\right|^2 (2\pi)^4\, \delta^4\left(\sum p_{fi} - \sum k_i\right). \tag{3.143}$$

Again, we use the localisation to replace $\sum k_i \to p_i = p_A + p_B$. Further, $\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \left|f_{\mathbf{p}}(\mathbf{k})\right|^2 = 1$ and $\sum p_{fi} = \sum p_f - p_i$. Finally, we get the

differential cross section

$$\mathrm{d}\sigma = \frac{1}{4p_A^0\, p_B^0\, |v_A - v_B|} \left|M_{fi}\right|^2 (2\pi)^4 \delta^4(p_f - p_i) \prod_{i=1}^{n} \frac{\mathrm{d}^3 p_{fi}}{(2\pi)^3} \frac{1}{2p_{fi}^0}. \tag{3.144}$$

Note, that except for the first fraction all expressions are Lorentz invariant. The first fraction is invariant under boosts along the beam axis. Thus, $\mathrm{d}\sigma$ is a (differential) transverse area, i.e. **invariant under boosts along the beam axis**. We define the n-particle phase space factor as

$$\mathrm{d}\Pi_n := (2\pi)^4 \delta^4(p_f - p_i) \prod_{i=1}^{n} \frac{\mathrm{d}^3 p_{fi}}{(2\pi)^3} \frac{1}{2p_{fi}^0}. \tag{3.145}$$

Let us now consider the highly relativistic case. Then

$$\begin{aligned} |s| &= (p_A + p_B)^2 = \left(p_A^0\right)^2 - \mathbf{p_A}^2 + \left(p_B^0\right)^2 - \mathbf{p_B}^2 + 2\, p_A^0\, p_B^0 - 2\, \mathbf{p_A}\, \mathbf{p_B} \\ &= m_A^2 + m_B^2 + 2\, p_A^0\, p_B^0 - 2\, \mathbf{p_A}\, \mathbf{p_B} \gg m_A^2 + m_B^2 . \\ &\Rightarrow 4p_A^0\, p_B^0\, |v_A - v_B| \to 2s\,, \end{aligned} \tag{3.146}$$

and

$$\mathrm{d}\sigma = \frac{1}{2s} \left|M_{fi}\right|^2 \mathrm{d}\Pi_n . \tag{3.147}$$

Note, that Eq. (3.147) can be rewritten in a manifestly boost-invariant way:

$$\mathrm{d}\sigma = \frac{1}{2w(s, m_A^2, m_B^2)} \left|M_{fi}\right|^2 \mathrm{d}\Pi_n\,, \tag{3.148}$$

with $w(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2xy - 2xz - 2yz}$.
Let us exemplary discuss the 2-2 scattering in ϕ^4-theory in the highly relativistic case. Then, we have $n = 2$ in Eq. (3.147) and $p_{fi} = p_i$. It follows

$$\begin{aligned} \int \mathrm{d}\Pi_2 &= \int (2\pi)^4\, \delta^4\Big(p_1 + p_2 - (p_A + p_B)\Big) \cdot \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \frac{1}{2p_1^0} \frac{\mathrm{d}^3 p_2}{(2\pi)^3} \frac{1}{2p_2^0} \\ &\simeq \frac{1}{(2\pi)^2\, 4p_1^0\, p_2^0} \int \mathrm{d}^3 p_2\, \delta\left(p_1^0 + p_2^0 - \sqrt{s}\right) \quad \text{for} \quad (p_A + p_B)^2 \gg m_A^2, m_B^2 . \end{aligned} \tag{3.149}$$

We compute this in the center of mass system (CMS). Therefore, we have $\mathbf{p_1} = -\mathbf{p_2} \Rightarrow p_1^0 = p_2^0$, i.e. equal masses. We also use

$$\mathrm{d}^3 p_2 = \mathrm{d}\Omega\, |\mathbf{p_2}|^2 \mathrm{d}|\mathbf{p_2}|\,, \tag{3.150}$$

with the solid angle $\mathrm{d}\Omega = \mathrm{d}\varphi \sin\theta\, \mathrm{d}\theta$.
It follows, $p_1^0 + p_2^0 - \sqrt{s} \simeq 2p_2^0 - \sqrt{s} = 2|\mathbf{p_2}|^2 - \sqrt{s}$, $p_i^0 = \sqrt{s}/2$

$$\int \mathrm{d}\Pi_2 = \frac{1}{2} \frac{s/4}{2(\pi)^2\, 4p_1^0\, p_2^0}\, \mathrm{d}\Omega = \frac{1}{32\pi^2} \mathrm{d}\Omega\,. \tag{3.151}$$

Now we use Eq. (3.40), i.e. that for classical scattering it is

$$\left|M_{fi}\right|^2 = \lambda^2\,. \tag{3.152}$$

With this, we obtain the

differential cross section (2-2 scattering)

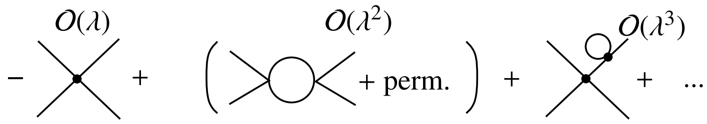
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2s} \left|M_{fi}\right|^2 \int\limits_{\mathrm{d}\Omega(p_2)\mathrm{fixed}} \mathrm{d}\Pi_2 = \frac{\lambda^2}{64\pi^2\, s}\,. \tag{3.153}$$

Lastly, we discuss the **computation of the S-matrix elements**. In the 2-2 scattering example we used, that

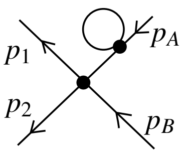
$$\left|M_{fi}\right|^2 = \lambda^2 + O(\lambda^3)\,. \tag{3.154}$$

We make an expansion in the Feynman diagrams:

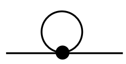
$\langle \mathbf{p_1}\, \mathbf{p_2} |\, \mathrm{i}\, T\, | \mathbf{p_A}\, \mathbf{p_B} \rangle$



Computing



with

$\frac{1}{2}$  $= -\mathrm{i}\, \Pi$

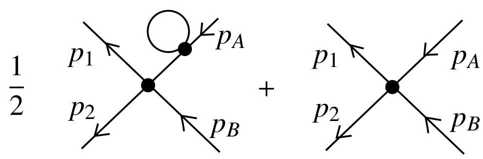
gives

$$\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon} (-\mathrm{i}\, \Pi)(-\mathrm{i}\lambda) \cdot (2\pi)^4\, \delta^4(p_A + p_B - p_1 - p_2)$$

$$= \left(\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\right)^{-1} \frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon} (-\mathrm{i}\, \Pi)\, \frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon} (-\mathrm{i}\lambda) \cdot ...$$

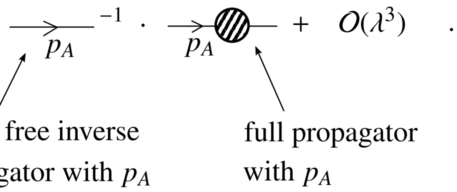
$$... \cdot \delta^4(p_A + p_B - p_1 - p_2)$$

$\Rightarrow$



$$
\begin{aligned}
&= -\mathrm{i}\lambda \left(\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\right)^{-1} \left(\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon} + \frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}(-\mathrm{i}\,\Pi)\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\right) \cdot ... \\
&\quad ... \cdot \delta^4(p_A + p_B - p_1 - p_2) \\
&= -\mathrm{i}\lambda \left(\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\right)^{-1} \frac{\mathrm{i}}{p_A - (m^2 + \Pi) + \mathrm{i}\epsilon} \cdot \delta^4(p_A + p_B - p_1 - p_2) \qquad (3.155)
\end{aligned}
$$

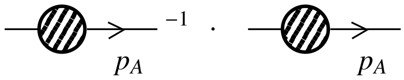
$= \left[-\mathrm{i}\lambda\, \delta^4(p_A + p_B - p_1 - p_2)\right]$



We remark, that the free inverse propagator is related to the fact, that the particle A in the initial state was prepared as a free state, which is only true for $t \to -\infty$. The correct state should relate to full (inverse) propagation, i.e.

This leads to

$1 \quad =$



in the above equation. Thus, we conclude, thatM_{fi} **is computed by computing amputated, connected scattering diagrams**. This will be discussed further in the subsequent section.

V. LSZ-Formalism

In this section we aim to compute the elements of the S-matrix. We will derive, that this can be done with the LSZ-reduction formula, named after the three German physicists Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann. In the previous section we have seen, that the naive preparation of our in-state lead to a product of the free inverse propagator with the full propagator in our scattering amplitudes (Eq. (3.155)). We have encountered a similar problem with vacuum bubbles before. In this section we shall see that

$$\phi_H(t \to \mp\infty) \to Z^{1/2}\, \phi_{\text{in/out}} \quad \text{(weak op. equivalence)}\,, \qquad (3.156)$$

with $Z \leq 1$. So far, we have implicitly assumed $Z = 1$. In the following we determine Z by computing the two-point function and subsequent generalisation to the n-point function. We begin with the vacuum expectation value of the two-point function

$$
\begin{aligned}
\langle \phi_H(x)\, \phi_H(y) \rangle &= \langle \Omega | \phi_H(x)\, \phi_H(y) | \Omega \rangle \\
&= \sum\!\!\!\!\!\!\int_{\lambda,\mathbf{p}} \langle \Omega | \phi_H(x)\ |\lambda, \mathbf{p}\rangle_H \, \langle \lambda, \mathbf{p}|_H \ \phi_H(y) | \Omega \rangle\ . \qquad (3.157)
\end{aligned}
$$

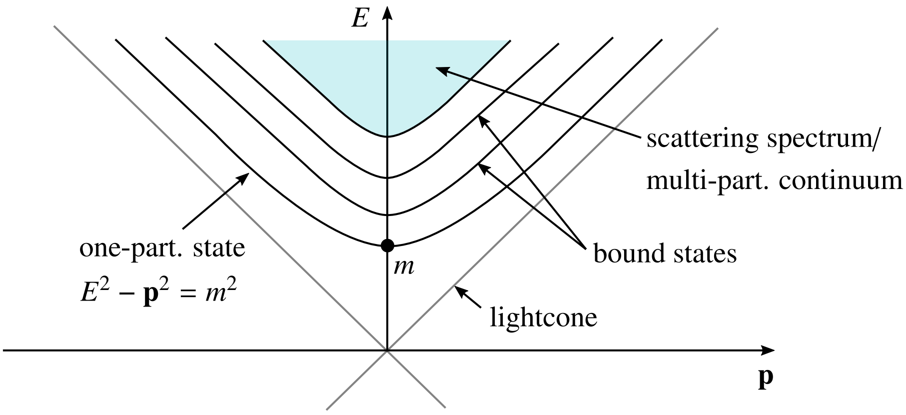
Where we chose $|\lambda, \mathbf{p}\rangle_H$ as eigenstates of H, i.e.

$$H\,|\lambda, \mathbf{p}\rangle = E_\lambda\;|\lambda, \mathbf{p}\rangle \tag{3.158}$$

and

$$\mathbf{P}\,|\lambda, \mathbf{p}\rangle = \mathbf{p}_\lambda\;|\lambda, \mathbf{p}\rangle\;, \tag{3.159}$$

with $E_\lambda^2 - \mathbf{p}_\lambda^2 = m_\lambda^2$ fixed (on-shell). The idea for the next steps is to integrate over all states (on-shell) with different masses to obtain the representation of the two-point function, which then will be off-shell. The fixed states $|\lambda, \mathbf{p}\rangle$ with fixed m_λ are connected by boosts.



Note, that for the vacuum state it is

$$E_\lambda = 0\;:\;|\Omega\rangle\;. \tag{3.160}$$

Furthermore,

$$\begin{aligned}\mathbb{1} &= |\Omega\rangle\,\langle\Omega| + \sum\!\!\!\!\!\!\int_\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_\lambda(\mathbf{p})}\,|\lambda, \mathbf{p}\rangle\,\langle\lambda, \mathbf{p}| \\ \langle\Omega|\phi(x)|\Omega\rangle &= 0\,. \end{aligned}\tag{3.161}$$

We use, that with $\hat{P} = (H, \mathbf{P})$ we have

$$\phi(x) = \mathrm{e}^{\mathrm{i}\hat{P}x}\,\phi(0)\,\mathrm{e}^{-\mathrm{i}\hat{P}x}\,. \tag{3.162}$$

Then we get, $(x^0 \geq y^0)$

$$\begin{aligned}\langle\phi(x)\,\phi(y)\rangle &= \sum\!\!\!\!\!\!\int_\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_\lambda(\mathbf{p})}\Big|\langle\Omega|\,\phi(0)\,|\lambda, \mathbf{p}\rangle\Big|^2 \cdot \mathrm{e}^{-\mathrm{i}\,p_\lambda(x-y)} \\ \text{using Eq. (2.21)} \quad \to \quad &= \sum\!\!\!\!\!\!\int_\lambda \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{p^2 - m_\lambda^2 + \mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}\,p(x-y)} \cdot \Big|\langle\Omega|\,\phi(0)\,|\lambda, \mathbf{p}\rangle\Big|^2. \end{aligned}\tag{3.163}$$

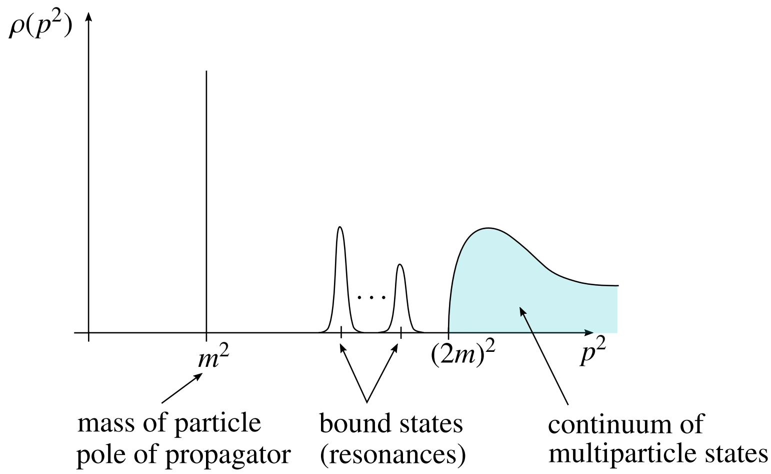


Figure 3.5.: Sketch of the spectral function.

Using the same steps for $x^0 \leq y^0$, we obtain in summary the

Källén-Lehmann spectral representation

$$\langle T\, \phi(x)\, \phi(y) \rangle = \int_0^\infty \frac{\mathrm{d}M^2}{2\pi}\, \rho(M^2)\, \mathcal{D}_F(x-y; M^2)\,, \tag{3.164}$$

with the

spectral function

$$\rho(p^2) = \sum\!\!\!\!\!\!\int_\lambda (2\pi)\, \delta(p^2 - m_\lambda^2) \left| \langle \Omega | \, \phi(0) \, | \lambda \rangle \right|^2 . \tag{3.165}$$

The spectral function is depicted in figure 3.5 and has the representation

$$\rho(p^2) = Z \cdot 2\pi\, \delta(p^2 - m^2) + \theta(p^2 - m_1^2) + \cdots\,, \tag{3.166}$$

where m_1^2 denotes the mass in the first residue. Hence,

$$\langle T\, \phi(x)\, \phi(y) \rangle = Z\, \mathcal{D}_F(x-y; m^2) + \int_{m_1^2}^\infty \frac{\mathrm{d}M^2}{2\pi}\, \rho(M^2)\, \mathcal{D}_F(x-y; M^2)\,. \tag{3.167}$$

Note, that $\mathcal{D}_F(x-y; M^2)$ carries the one-particle pole of ϕ.
To relate this to ϕ_{in}, we consider one-particle states $|\lambda_1\rangle$ in $|\lambda\rangle\langle\lambda|$:

$$\rho \sim \sum\!\!\!\!\!\!\int_{\substack{\text{one-part.}\\ \text{states}\lambda_1}} \mathrm{e}^{-\mathrm{i}p_\lambda(x-y)} \left| \langle \Omega | \, \phi(0) \, | \lambda_1 \rangle \right|^2 , \tag{3.168}$$

with, $U = U(-\infty, 0)$

$$\begin{aligned} \left| \langle \Omega | \, \phi(0) \, | \lambda_1 \rangle \right|^2 &= \left| \langle \Omega | \, U^{-1}\, U\, \phi\, U^{-1}\, U \, | \lambda_1 \rangle \right|^2 \\ &= \left| \langle 0 | \, Z^{1/2}\, \phi_{\text{in}} \, | \lambda_1 \rangle_I \right|^2 \\ &= Z\,. \end{aligned} \tag{3.169}$$

Let us now determine Z. For this purpose, we consider the *not time ordered* expectation value $\langle\phi(x)\,\phi(y)\rangle$. Then $\mathcal{D}_F$ in Eq. (3.167) is substituted by the not time ordered propagator D. Note also, that

$$\begin{aligned}\left[\frac{\partial}{\partial y^0}\,\langle[\phi(x),\phi(y)]\rangle\right]_{x^0=y^0} &= \left\langle[\phi(x),\Pi(y)]_{x^0=y^0}\right\rangle \\ &= \mathrm{i}\,\delta^3(\mathbf{x}-\mathbf{y}) \qquad (3.170)\end{aligned}$$

and

$$\left[\frac{\partial}{\partial y^0}\Big(D(x-y)-D(y-x)\Big)\right]_{x^0=y^0} = \mathrm{i}\,\delta^3(\mathbf{x}-\mathbf{y})\,. \qquad (3.171)$$

Next, we integrate over space, i.e. evaluate $\int \mathrm{d}^3x\,\left\langle[\phi(x),\Pi(y)]_{x^0=y^0}\right\rangle$. With Eq. (3.167) and Eq. (3.170) we obtain

$$1 = Z + \int_{m_1^2}^{\infty} \frac{\mathrm{d}M^2}{2\pi}\,\rho(M^2) \qquad (3.172)$$

and, as the integral term is larger than zero,

$$0 \le Z \le 1\,. \qquad (3.173)$$

Note, that $Z=1$ in free theory and $Z<1$ in interacting theory. Also note, that $1-Z$ accounts for the overlap of $\phi\,|\Omega\rangle$ with multi-particle states and that in the limit $t\to\mp\infty$:

$$\phi(x) \to Z^{1/2}\,\phi_{\text{in/out}} \quad \text{(weak op. equivalence)}\,. \qquad (3.174)$$

The propagator on-shell is

$$\begin{aligned}\mathcal{D}_F(p^2\to m^2) &\simeq \frac{\mathrm{i}\,Z}{p^2-m^2+\mathrm{i}\epsilon} \qquad (3.175)\\ \Big(&= \int \mathrm{d}^4x\;\mathrm{e}^{\mathrm{i}px}\,\langle T\;\phi(x)\phi(0)\rangle\Big),\end{aligned}$$

where m^2 is *not* simply the mass parameter m_0^2 in the Lagrangian.
Now we will derive the **LSZ-reduction formula**. For this purpose we extend the analysis of the two-point function to an n-point function. The latter will be related to the S-matrix elements. As in Eq. (3.175) we evaluate the Fourier transform

$$\int \mathrm{d}^4x\;\mathrm{e}^{\mathrm{i}px}\,\langle T\;\phi(x)\,\phi(x_2)\cdots\phi(x_n)\rangle\;. \qquad (3.176)$$

With $T_+ > x_2^0,\ldots,x_n^0$ and $T_- < x_2^0,\ldots,x_n^0$, we split

$$\int \mathrm{d}x^0\;\mathrm{e}^{\mathrm{i}p^0x^0} = \left(\int_{-\infty}^{T_-} + \int_{T_-}^{T_+} + \int_{T_+}^{+\infty}\right)\mathrm{d}x^0\;\mathrm{e}^{\mathrm{i}p^0x^0}\,, \qquad (3.177)$$

where the first and the third integral give poles and the second one is finite. It follows

$$\begin{aligned}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \langle T\, \phi(x)\,\phi(x_2)\cdots\phi(x_n)\rangle &= \int_{T_+}^{\infty} \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \langle \phi(x) T\, \phi(x_2)\cdots\phi(x_n)\rangle + \left(\int_{-\infty}^{T_-} + \int_{T_-}^{T_+}\right)\cdots \\ &= \int_{T_+}^{\infty} \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \sum\!\!\!\!\!\!\int_{\lambda} \int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{q}}}\cdot \ldots \\ &\qquad \ldots\cdot \langle \phi(x)|\lambda,\mathbf{p}\rangle \langle \lambda,\mathbf{p}|T\, \phi_2\cdots\phi_n\rangle + \cdots\,. \end{aligned} \tag{3.178}$$

Using $\langle \phi(x)|\lambda,\mathbf{p}\rangle = \langle \Omega|\,\phi(0)\,|\lambda\rangle\ \mathrm{e}^{-\mathrm{i}qx}$:

$$\begin{aligned} &\sum\!\!\!\!\!\!\int_{\lambda} \int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{2\omega\mathbf{q}} \mathrm{d}x^0\ \mathrm{e}^{\mathrm{i}(p^0-q^0+\mathrm{i}\epsilon)x^0}\ \langle\Omega|\,\phi(0)\,|\lambda\rangle \langle\lambda,\mathbf{p}|T\,\phi_2\cdots\phi_n\rangle\cdot(2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{q}) \\ = &\sum\!\!\!\!\!\!\int \frac{1}{2\omega_{\mathbf{p}}} \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}(p^0-\omega_{\mathbf{p}}+\mathrm{i}\epsilon)x^0}T_+}{p^0-\omega_{\mathbf{p}}+\mathrm{i}\epsilon}\ \langle\Omega|\,\phi(0)\,|\lambda\rangle \langle\lambda,\mathbf{p}|\phi_2\cdots\phi_n\rangle\,. \end{aligned} \tag{3.179}$$

For $p^0 \to \omega_{\mathbf{p}}$: (using Källén-Lehmann)

$$\lim_{p^0\to\omega_{\mathbf{p}}} \int_{T_+}^{+\infty} \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px}\ \langle T\ \phi(x)\,\phi(x_2)\cdots\phi(x_n)\rangle = \frac{\mathrm{i}\,Z^{1/2}}{p^2-m^2+\mathrm{i}\epsilon}\langle\mathbf{p}|T\ \phi(x_2)\cdots\phi(x_n)\rangle + \text{finite}\,. \tag{3.180}$$

Analogously we find for the $\int_{-\infty}^{T_-}$-term:

$$\lim_{p^0\to-\omega_{\mathbf{p}}} \int_{-\infty}^{T_-} \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \Big\langle \Big(T\ \phi(x_2)\cdots\phi(x_n)\Big)\phi(x)\Big\rangle = \frac{\mathrm{i}\,Z^{1/2}}{p^2-m^2+\mathrm{i}\epsilon}\langle T\ \phi(x_2)\cdots\phi(x_n)|{-\mathbf{p}}\rangle + \text{finite}\,. \tag{3.181}$$

As mentioned before, the last term $\int_{T_-} T_+ \cdots$ is finite as the integration interval has a finite length (compact). We remark, that the above analysis can be repeated iteratively for all $\phi(x_i)$. Strictly speaking, one should separate the fields spacially: $\int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}px} \to \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}px}\, \mathrm{f}_{\mathbf{p}}(\mathbf{k})$. Further note, that states $|\mathbf{p}\rangle$ are at time $t\to\infty$ and states $\langle\mathbf{p}|$ are at time $t\to+\infty$, and that after iteration we have

$${}_{-\infty}\langle \mathbf{p_1}\cdots\mathbf{p_n}|\mathbf{k_1}\cdots\mathbf{k_m}\rangle_{+\infty} = \langle \mathbf{p_1}\cdots\mathbf{p_n}|\,S\,|\mathbf{k_1}\cdots\mathbf{k_m}\rangle\,. \tag{3.182}$$

With this we obtain the

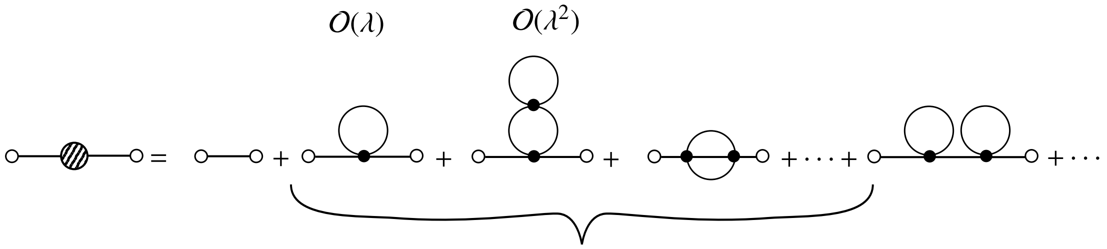
LSZ-reduction formula

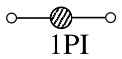
$$\begin{aligned} &\langle \mathbf{p_1}\cdots\mathbf{p_n}|\,S\,|\mathbf{k_1}\cdots\mathbf{k_m}\rangle\Big|_{\text{on-shell}} \\ = &\int \prod_{i=1}^{n}\mathrm{d}^4x_i\ \mathrm{e}^{\mathrm{i}p_ix_i} \prod_{j=1}^{m}\mathrm{d}^4y_i\ \mathrm{e}^{-\mathrm{i}k_jy_j} \prod_{i=1}^{n}\left(\partial_{x_i}^2+m^2\right)\prod_{j=1}^{m}\left(\partial_{y_i}^2+m^2\right)\cdot\ldots \\ &\ldots\cdot Z^{(n+m)/2}\,\langle T\ \phi(x_1)\cdots\phi(x_n)\,\phi(y_1)\cdots\phi(y_n)\rangle\,, \end{aligned} \tag{3.183}$$

where on-shell means $p^2 = m^2$ being the physical mass pole and not the mass parameter m_0^2 in the

Lagrangian. Let us now consider the structure of $\langle T\ \phi_1 \cdots \phi_n \rangle$.

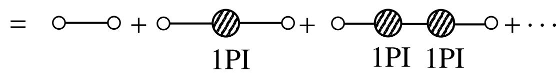
Example 8: n=2.



$\Pi(p)$: 

1PI: one-particle irreducible

cannot be split by cutting one line

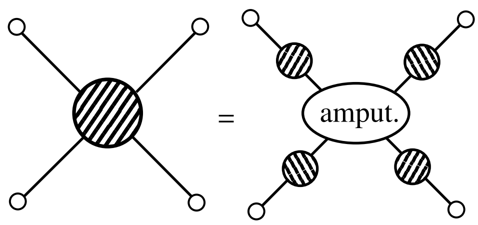


$$
\begin{aligned}
&= \frac{\mathrm{i}}{p^2 - m_0^2 + \mathrm{i}\epsilon} + \frac{\mathrm{i}}{p^2 - m_0^2 + \mathrm{i}\epsilon}\Big(-\mathrm{i}\,\Pi(p)\Big)\frac{\mathrm{i}}{p^2 - m_0^2 + \mathrm{i}\epsilon} + \cdots \\
&= \frac{\mathrm{i}}{p^2 - \Big(m_0^2 + \Pi(p)\Big) + \mathrm{i}\epsilon}
\end{aligned}
\tag{3.184}
$$

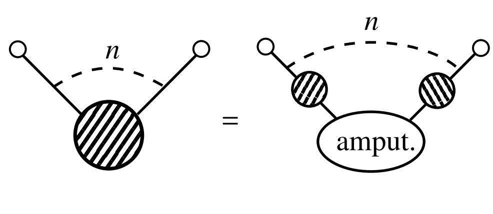
with

$$
\frac{\mathrm{i}}{p^2 - \Big(m_0^2 + \Pi(p)\Big) + \mathrm{i}\epsilon} \overset{p^2 \to m^2}{\longrightarrow} \frac{\mathrm{i}\,Z}{p^2 - m^2 + \mathrm{i}\epsilon}
\tag{3.185}
$$

Example 9: n=4.

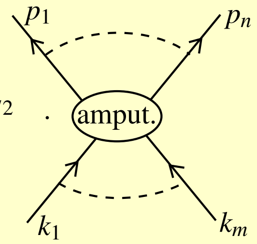


And in general:



This entails for the S-matrix elements with Eq. (3.183)

$$\langle \mathbf{p_1} \cdots \mathbf{p_n} |\, S \,| \mathbf{k_1} \cdots \mathbf{k_n} \rangle \Big|_{\text{on-shell}} = Z^{(n+m)/2} \quad \cdot \tag{3.186}$$



Concluding, we remark: Z is called wave function (or field strength) renormalisation, as it multiplies the field. Note that

$$\left\langle T\, Z^{-1/2}\,\phi(x)\, Z^{-1/2}\,\phi(y) \right\rangle_{p^2 \to m^2} = \mathcal{D}_F(x-y; m^2) \tag{3.187}$$

Z renormalises the field. With this we also see, that

$$\begin{aligned} & Z^{(n+m)/2}\, \langle T\, \phi(p_1) \cdots \phi(k_m) \rangle_{amput.} \\ \cong\ & Z^{(n+m)/2} \prod_i \frac{p_i^2 + m^2}{Z^{1/2}} \prod_j \frac{k_j^2 + m^2}{Z^{1/2}}\, \langle T\, \phi(p_1) \cdots \phi(k_m) \rangle \\ =\ & \prod_i (p_i^2 + m^2) \prod_j (k_j^2 + m^2) \left\langle T\, Z^{-1/2}\,\phi(p_1) \cdots Z^{-1/2}\,\phi(k_m) \right\rangle , \end{aligned} \tag{3.188}$$

where $\left\langle T\, Z^{-1/2}\,\phi(p_1) \cdots Z^{-1/2}\,\phi(k_m) \right\rangle$ is just the expectation value of the renormalised fields.

4. Fermions

I. Fields and Lorentz Invariance

So far we have discussed the quantisation of a scalar field, i.e. particles with spin zero (Higgs boson). The scalar field is invariant under Lorentz transformations:

$$\phi(x) \xrightarrow{\Lambda} \phi'(x') = \phi(x)\,, \tag{4.1}$$

with

$$x^\mu \quad \to \quad (x')^\mu = \Lambda^\mu{}_\nu\, x^\nu \tag{4.2}$$

and $\phi'(x) = \phi(\Lambda^{-1}\, x)$. However, for vector fields we have

$$\begin{aligned} A^\mu(x) &\quad\to\quad \Lambda^\mu{}_\nu\, A^\nu(x) \\ (A')^\mu(x) &\quad=\quad \Lambda^\mu{}_\nu\, A^\nu(\Lambda^{-1}\, x)\,, \end{aligned} \tag{4.3}$$

and for Tensor fields (e.g. the fieldstrength in QED, QCD, week):

$$F^{\mu\nu}(x) \quad\to\quad \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\, F^{\rho\sigma}(x)\,. \tag{4.4}$$

Note, that the graviton is an example for a particle with spin two.
In the following we will present some mathematical background on group theory, as it is important to understand the properties of fermions. In general we can write

$$\phi^i(x) \quad\to\quad R(\Lambda)^i{}_j\, \phi^j(x)\,, \tag{4.5}$$

with the general index i, e.g. $i = \{\}, \mu, \mu\nu, \ldots$ and the representation R. The representation is chosen accordingly to the field, i.e.

$$\begin{aligned} \text{scalar:} \quad R(\Lambda) &= 1 \quad \textit{trivial representation} \\ \text{vector:} \quad R(\Lambda) &= \Lambda \quad \textit{fundamental representation} \\ \text{(2nd rank) tensor:} \quad R(\Lambda) &= \left(\Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\right) \quad \textit{tensor representation}\,. \end{aligned} \tag{4.6}$$

Let G denote a group. Then, the representation $R:\; G \to R(G)$ has the properties:

$$\begin{aligned} R(\mathbb{1}) &= \mathbb{1}\,, \\ R(g\cdot h) &= R(g)\cdot R(h)\,. \end{aligned} \tag{4.7}$$

For instance, for rotations in $\mathbb{R}^3$, i.e. the SO(3) group we have

$$\begin{aligned} \text{trivial rep:} \quad R(\Lambda) &= 1 \quad \Lambda \in \mathrm{SO}(3) \\ \text{fundamental rep:} \quad R(\Lambda) &= \Lambda \quad \text{Lie group}\,. \end{aligned} \tag{4.8}$$

A useful property of Lie groups is, that we can write every element in terms of an exponential

$$\Lambda = \mathrm{e}^{\mathrm{i}\omega \mathbf{J}}\,, \tag{4.9}$$

where ω is a vector and $\mathbf{J}$ is a Lie-algebra with the generators J^i. These generate infinitesimal rotations with axis x^i: $(\partial^i x^j = -\delta^i{}_j)$

$$\begin{aligned} J^i &= -\mathrm{i}\epsilon^{ijk}\, x^j \partial^k \\ &= -\frac{1}{2}\epsilon^{ijk} J^{jk}\,, \quad J^{jk} = -\mathrm{i}\left(x^j\partial^k - x^k\partial^j\right)\,. \end{aligned} \tag{4.10}$$

Note, that this is for instance used in quantum mechanics in the n-dimensional representation of spins with $n = 2n + 1$. As this chapter is about fermions, we now consider the situation with spin $1/2$. Then, we have the generators $\frac{\sigma^i}{2}$, with

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = \mathrm{i}\epsilon^{ijk}\,\frac{\sigma^k}{2}\,, \tag{4.11}$$

where σ^i are the Pauli matrices (spinor representation)

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.12}$$

We remark, that the Lie algebra provides *local* information about the Lie group (tangential space). We will now present two important examples.

Example 10: SO(3) and SU(2)$\simeq$ S^3.

Lie algebra:

$$\left[t^a, t^b\right] = \mathrm{i}\epsilon^{abc}\, t^c\,. \tag{4.13}$$

As the Lie group is a differentiable manifold, the SU(2) is the double covering of SO(3)$\simeq$ RP^3, which is visualised in figure 4.1

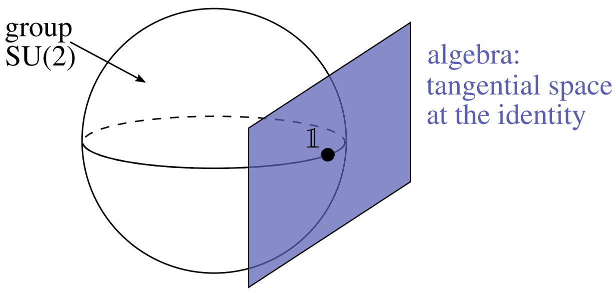


Figure 4.1.: Schematic representation of the SU(2) and the Lie algebra.

Example 11: SO(1,3) and SL (2,$\mathbb{C}$).

Consider the infinitesimal Lorentz transformation $\Lambda \in SO(1,3)$:

$$\Lambda_\mu^{\ \nu} =_\mu^{\ \nu} + \mathrm{i}\, T_\mu^{\ \nu} \,. \tag{4.14}$$

From

$$\Lambda_\mu^{\ \nu}\, \Lambda_\rho^{\ \sigma}\, \eta_{\nu\sigma} = \eta_{\mu\rho} \tag{4.15}$$

follows:

$$\begin{aligned} \left(\delta_\mu^{\ \nu} + \mathrm{i}\, T_\mu^{\ \nu}\right)\left(\delta_\rho^{\ \sigma} + \mathrm{i}\, T_\rho^{\ \sigma}\right)\eta_{\nu\sigma} &= \eta_{\mu\rho} + O(T^2) \\ \Rightarrow\ T_{\mu\rho} + T_{\rho\mu} &= 0\,. \end{aligned} \tag{4.16}$$

We conclude, that T has $\frac{16-4}{2} = 6$ free parts, of which three are given by boosts and the other three by the

generators M

$$T_\mu^{\ \nu} = \frac{\omega^{\rho\sigma}}{2}\left(M_{\rho\sigma}\right)_\mu^{\ \nu} \,, \tag{4.17}$$

with

$$[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}\,(\eta^{\nu\rho}\, M^{\mu\sigma} - \eta^{\mu\rho}\, M^{\nu\sigma} - \eta^{\nu\sigma}\, M^{\mu\rho} + \eta^{\mu\sigma}\, M^{\nu\rho}) \,. \tag{4.18}$$

Eq. (4.18) is the Lie algebra of SO(1,3) rotations. To see this, we extend the SO(3)-generators of rotations, J^{ij} in Eq. (4.10) to boosts (J^{0i}) and find

$$J^{\mu\nu} = \mathrm{i}\,(x^\mu \partial^n u - x^\nu \partial^\mu) \,, \tag{4.19}$$

which satisfy Eq. (4.18). To find general representations, we also look for M, that satisfy Eq. (4.18). For the fundamental representation we obtain for example

$$(M^{\mu\nu})_{\rho\sigma} = \mathrm{i}\left(\delta^\mu{}_\rho\, \delta^\nu{}_\sigma - \delta^\mu{}_\sigma\, \delta^\nu{}_\rho\right) \,. \tag{4.20}$$

Thus, boosts and rotations are given by

$$\begin{aligned} J_i &= \frac{1}{2}\epsilon_{ijk}\, M_{jk} \quad \textit{rotations} \\ K_i &= M_{0i} \quad \textit{boosts}\,. \end{aligned} \tag{4.21}$$

Example 12: Boosts along $\mathbf{x_1}$-axis.

$$\begin{aligned} \Lambda^\mu{}_\nu &= \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-v^2}} \\ &= \left(\mathrm{e}^{w\, k_1}\right)^\mu{}_\nu \,, \end{aligned} \tag{4.22}$$

with the rapidity $w = \text{arctanh}\frac{v}{2}$ and the generator

$$K_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.23}$$

The generators in Eq. (4.21) make the structure of the Lorentz group apparent, as we can now formulate the Lie-algebra in terms of **J** and **K**

$$\begin{aligned} \left[J^i, J^j\right] &= \mathrm{i}\epsilon^{ijk} J^k \\ \left[K^i, K^j\right] &= -\mathrm{i}\epsilon^{ijk} J^k \\ \left[J^i, K^j\right] &= \mathrm{i}\epsilon^{ijk} K^k . \end{aligned} \tag{4.24}$$

We remark, that SU(2) and SL(2,$\mathbb{C}$) with the generators ($J^i + \mathrm{i}K^i, J^i - \mathrm{i}K^i$) are the universal covering groups of SO(3) and SO(1,3), respectively. Note: For a universal covering group $\tilde{G}$ of G it holds:

$$'\text{simply connected group } \tilde{G} \supseteq G' . \tag{4.25}$$

II. Spinor Fields

In section I we have discussed the mathematical structure of the Lorentz group. Let us now use these concepts to describe spinor fields. For this purpose we combine boosts K^i and rotations J^i into

$$\begin{aligned} N^i &= \frac{1}{2}\left(J^i + \mathrm{i}\,K^i\right) \\ (N^i)^\dagger &= \frac{1}{2}\left(J^i - \mathrm{i}\,K^i\right) . \end{aligned} \tag{4.26}$$

The N's and $N^\dagger$'s have the SO(3) i.e. SU(2) Lie-algebra:

$$\left[N_i^{(\dagger)}, N_j^{(\dagger)}\right] = \mathrm{i}\,\epsilon_{ijk}\,N_k^{(\dagger)} . \tag{4.27}$$

Hence, we can formulate a two-dimensional spin 1/2 representation:

$$\begin{aligned} \text{left-handed:} \quad \Lambda_L &= \exp\left(\frac{\mathrm{i}}{2}\,\sigma^i\,(w_i - \mathrm{i}\,v_i)\right) \\ \text{right-handed:} \quad \Lambda_R &= \exp\left\{\frac{\mathrm{i}}{2}\,\sigma^i\,(w_i + \mathrm{i}\,v_i)\right\}, \end{aligned} \tag{4.28}$$

where w_i and v_i denote rotations and boosts, respectively and Λ_L, $\Lambda_R \in$ SL(2,$\mathbb{C}$)[1]. Note, that under parity transformations

$$\begin{aligned} \left(x^0, \mathbf{x}\right) &\xrightarrow{P} \left(x^0, -\mathbf{x}\right) \\ \Rightarrow \mathbf{J} &\xrightarrow{P} \mathbf{J} \quad \textit{pseudo-vector} \\ \Rightarrow \mathbf{K} &\xrightarrow{P} -\mathbf{K} \quad \textit{vector}. \end{aligned} \tag{4.29}$$

[1] universal covering group of the Lorentz group

To determine, how $\Lambda_{L/R}$ act on coordinates, we define

$$\hat{x} = x_\mu \sigma^\mu \,, \tag{4.30}$$

with

$$(\sigma^\mu) = \left(\sigma^0, \sigma^1, \sigma^2, \sigma^3\right) \,, \quad \sigma^0 = \mathbb{1}_{2\times 2} \,, \tag{4.31}$$

and the Pauli matrices $\sigma^{1,2,3}$. Then:

$$\hat{x} = \begin{pmatrix} x_0 - x_3 & x_1 + \mathrm{i}\, x_2 \\ x_1 - \mathrm{i}\, x_2 & x_0 + x_3 \end{pmatrix} , \tag{4.32}$$

and

$$\det \hat{x} = x_\mu x^\mu \,. \tag{4.33}$$

Note, that Lorentz transformations leave the determinant unchanged

$$\hat{x}' = \Lambda_L \, \hat{x} \, \Lambda_L^\dagger \quad \text{with} \quad \det \hat{x}' = \det \hat{x} \,, \tag{4.34}$$

as $\det \Lambda_L^{(\dagger)} = 1$. We remark, that Λ_L and $-\Lambda_L$ give the same $\hat{x}'$ (double covering). Further

$$\Lambda_{L/R}^\dagger = \Lambda_{R/L}^{-1} \,, \tag{4.35}$$

and σ maps $L \to R$. Also note, that σ^μ transforms as a vector. We can now formulate the field equations for a two-component spinor

$$\mathcal{D}_L \Psi_L = 0 \,, \tag{4.36}$$

where Ψ_L is the left-handed *Weyl spinor*. Under Lorentz transformation, it holds

$$\begin{aligned} \Psi_L(x) &\xrightarrow{\Lambda} \Lambda_L \Psi_L(x) \\ \mathcal{D}_L \Psi_L(x) &\xrightarrow{\Lambda} \mathcal{D}'_L \Lambda_L \Psi_L(x) = \Lambda_R \mathcal{D}_L \Psi_L(x) \\ \Rightarrow \mathcal{D}'_L &= \Lambda_R \mathcal{D}_L \Lambda_R^\dagger \,, \quad \text{as} \quad \Lambda_R^\dagger = \Lambda_L^{-1} \,, \end{aligned} \tag{4.37}$$

with $\mathcal{D}_L = \mathrm{i}\, \bar{\sigma}^\mu \partial_\mu$ and $\bar{\sigma} = \left(\sigma^0, -\vec{\sigma}\right)$. Analogously, this holds for the right-handed Weyl spinor, with $\mathcal{D}_R = \mathrm{i}\, \sigma^\mu \partial_\mu$, which yields the

Weyl equations

$$\begin{aligned} \mathrm{i}\, \bar{\sigma}^\mu \partial_\mu \, \Psi_L &= 0 \\ \mathrm{i}\, \sigma^\mu \partial_\mu \, \Psi_R &= 0 \,, \end{aligned} \tag{4.38}$$

which form the equations of motion of two-component spinors. Note, that the Weyl equations (4.38) do not have parity invariance.

Let us now connect Eq. (4.38) to the Klein-Gordon equation (2.14):

$$\begin{aligned} & \sigma^\mu \partial_\mu \left(\bar{\sigma}^\nu \partial_\nu \, \Psi_L = 0\right) \\ = \;& \frac{1}{2} \{\sigma^\mu, \bar{\sigma}^\nu\} \, \partial_\mu \partial_\nu \, \Psi_L = \frac{1}{2} \left(2\eta^{\mu\nu}\right) \partial_\mu \partial_\nu \, \Psi_L \\ \Rightarrow \;& \partial_\mu \partial^\mu \, \Psi_L = \partial^2 \, \Psi_L = 0 \,, \end{aligned} \tag{4.39}$$

where we used the anti-commutator

$$\{\sigma^\mu, \bar{\sigma}^\nu\} = \sigma^\mu \bar{\sigma}^\nu + \bar{\sigma}^\nu \sigma^\mu = 2\eta^{\mu\nu} . \tag{4.40}$$

Similarly, one shows $\partial^2 \Psi_R = 0$, which implies, that the Weyl spinors also satisfy the Klein-Gorden equation.
If we demand parity invariance, we have to combine left- and right-handed spinors. This gives the **Dirac spinor**

$$\Psi_D = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} , \tag{4.41}$$

which basically partitions space into the space of left- and right-handed spinors. Then, $\mathcal{D}_L$ maps left- to right-handed spinors and $\mathcal{D}_R$ maps right- to left-handed spinors. Now we combine the Weyl-operators $\mathcal{D}_{L/R}$

$$\begin{pmatrix} 0 & \mathcal{D}_R \\ \mathcal{D}_L & 0 \end{pmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \mathrm{i}\, \gamma^\mu \partial_\mu \Psi_D , \tag{4.42}$$

with the matrix γ^μ (chiral representation)

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} . \tag{4.43}$$

The γ matrices are Lorentz invariant and have the desired parity invariance. Further they satisfy the

Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} . \tag{4.44}$$

To perform Lorentz transformations, we now define the four-dimensional spin 1/2 representation of Λ:

$$\Lambda_{1/2} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} , \tag{4.45}$$

with

$$\begin{aligned} \Psi_D &\xrightarrow{\Lambda} \Lambda_{1/2}\, \Psi_D = \begin{pmatrix} \Lambda_L \Psi_L \\ \Lambda_R \Psi_R \end{pmatrix} \\ \mathrm{i}\, \gamma^\mu \partial_\mu \Psi_D &\xrightarrow{\Lambda} \Lambda_{1/2}\, \mathrm{i}\, \gamma^\mu \partial_\mu \Lambda_{1/2}^{-1} \Lambda_{1/2} \Psi_D = \Lambda_{1/2}\, \mathrm{i}\, \gamma^\mu \partial_\mu \Psi_D . \end{aligned} \tag{4.46}$$

With $\Psi = \Psi_D$, we can formulate the

Dirac equation

$$(\mathrm{i}\, \partial\!\!\!/ - m)\Psi = 0 , \tag{4.47}$$

with the short notation

$$w\!\!\!/ := \gamma^\mu w_\mu . \tag{4.48}$$

We can now write down the generators M (Eq. (4.17)) in spin-representation:

$$S^{\mu\nu} = \frac{\mathrm{i}}{4} [\gamma^\mu, \gamma^\nu] , \tag{4.49}$$

with

$$[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{pmatrix}$$

$$\begin{aligned} \sigma\bar{\sigma} &: L \to L \\ \bar{\sigma}\sigma &: R \to R \end{aligned} \tag{4.50}$$

Note, that (see Eq. (4.21))

$$\begin{aligned} K_{i_L} &= S_{0i_L} = -\mathrm{i}\,\sigma_{i/2} = \mathrm{i}\,\bar{\sigma}_{i/2} \\ J_{i_L} &= \frac{1}{2}\epsilon_{ijk} S_{jk} = -\frac{\mathrm{i}}{2}\epsilon_{ijk}\left[\sigma_{j/2}, \sigma_{k/2}\right] = -\frac{\mathrm{i}}{2}\epsilon_{ijk}\left(\mathrm{i}\,\epsilon_{jkl}\,\sigma_{l/2}\right) = \sigma_{i/2}\,. \end{aligned} \tag{4.51}$$

Analogously, it follows

$$\begin{aligned} K_{i_R} &= \mathrm{i}\,\sigma_{i/2} \\ J_{i_R} &= \sigma_{i/2}\,, \end{aligned} \tag{4.52}$$

and hence

$$\Lambda_{1/2} = \mathrm{e}^{\mathrm{i}\,w_{\mu\nu/2}\,S^{\mu\nu}} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}, \tag{4.53}$$

with (see Eq. (4.28))

$$\begin{aligned} \Lambda_L &= \exp\left(\mathrm{i}\,\frac{\sigma^i}{2}\,(w_i - \mathrm{i}\,v_i)\right) \\ \Lambda_R &= \exp\left(\mathrm{i}\,\frac{\sigma^i}{2}\,(w_i + \mathrm{i}\,v_i)\right), \end{aligned} \tag{4.54}$$

and $w_{0i} = v_i$, $w_{ij} = \epsilon_{ijk}\,w_k$.
Next, let us find the inverse of $\Lambda_{1/2}$. γ^0 is hermitian, i.e.

$$\left(\gamma^0\right)^2 = \mathbb{1}_{4\times4}, \quad \left(\gamma^0\right)^\dagger = \gamma^0\,. \tag{4.55}$$

On the other hand, γ^i is anti-hermitian:

$$\left(\gamma^i\right)^2 = -\mathbb{1}_{4\times4}, \quad \left(\gamma^i\right)^\dagger = -\gamma^i\,. \tag{4.56}$$

Note, that these properties are representation independent. From Eq. (4.43) (i.e. choosing chiral representation) it also follows

$$\boxed{\gamma^0 \left(\gamma^i\right)^\dagger \gamma^0 = \gamma^i\,,} \tag{4.57}$$

and we conclude

$$\gamma^0\,(S^{\mu\nu})^\dagger\,\gamma^0 = -S^{\mu\nu}\,. \tag{4.58}$$

Thus, the inverse Lorentz transformation for the four-dimensional spin 1/2 representation is given by

$$\gamma^0 \Lambda_{1/2}^{\dagger} \gamma^0 = \Lambda_{1/2}^{-1} . \tag{4.59}$$

Note, that also the Dirac spinor satisfies the Klein-Gordon-equation (2.14):

$$\begin{aligned}
& \left(-\mathrm{i}\,\gamma^\mu \partial_\mu - m\right)(\mathrm{i}\,\gamma^\nu \partial_\nu - m)\ \Psi = \left(\gamma^\mu\,\gamma^\nu \partial_\mu \partial_\nu + m^2\right)\Psi \\
= \quad & \left(\frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\} \partial_\mu \partial_\nu + m^2\right)\Psi = \left(\frac{1}{2}\left(2\eta^{\mu\nu}\right) \partial_\mu \partial_\nu + m^2\right)\Psi \\
\Rightarrow \quad & \left(\partial_\mu \partial^\mu + m^2\right)\Psi = 0 .
\end{aligned} \tag{4.60}$$

In the following, we consider **the Lagrangian and the Hamiltonian of the spinor field**. The Lagrangian transforms as a Lorentz scalar $\sim (\mathrm{i}\not{\partial} - m)\ \Psi$

$$\begin{aligned}
\mathcal{L} \quad &= \quad \bar{\Psi}\,(\mathrm{i}\not{\partial} - m)\ \Psi \\
&\xrightarrow{\Lambda} \quad \bar{\Psi}' \Lambda_{1/2}\,(\mathrm{i}\not{\partial} - m)\ \Psi \\
&\Rightarrow \quad \bar{\Psi}' = \bar{\Psi} \Lambda_{1/2}^{-1} ,
\end{aligned} \tag{4.61}$$

as the Lagrangian is Lorentz invariant. It follows that

$$\bar{\Psi} = \Psi^\dagger \gamma^0 , \tag{4.62}$$

which is called *Dirac conjugate*, with

$$\begin{aligned}
\bar{\Psi}' \quad &= \quad \Psi^\dagger \Lambda_{1/2}^\dagger \gamma^0 = \Psi^\dagger \gamma^0 \gamma^0 \Lambda_{1/2}^\dagger \gamma^0 \\
\text{Eq. (4.59)} \quad \rightarrow \quad &= \quad \bar{\Psi} \Lambda_{1/2}^{-1} .
\end{aligned} \tag{4.63}$$

The equation of motion is given by the Dirac equation (4.47)

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \bar{\Psi}} \quad &= \quad 0 = (\mathrm{i}\not{\partial} - m)\ \Psi \\
\frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \quad &= \quad 0 = \bar{\Psi}\left(\mathrm{i}\overleftarrow{\not{\partial}} - m\right) ,
\end{aligned} \tag{4.64}$$

with the short notation $f\,\overleftarrow{\partial_\mu} = -\partial_\mu f$. Then the Hamiltonian density is given by

$$\begin{aligned}
\mathcal{H} \quad &= \quad \Pi_\Psi \dot{\Psi} - \mathcal{L} = \mathrm{i}\,\bar{\Psi}\gamma^0 \dot{\Psi} - \mathcal{L} \\
&= \quad \Psi^\dagger \gamma^0 \left(-\mathrm{i}\,\vec{\gamma}\,\vec{\partial} + m\right)\Psi ,
\end{aligned} \tag{4.65}$$

where $\vec{\gamma}\,\vec{\partial} = \gamma^i \partial_i = \gamma^i \frac{\partial}{\partial x^i}$.
Let us now discuss some **invariants and general properties**. The derivations above made use of a specific representation of our spinors in left- and right-handed Weyl spinors. In particular for massive Dirac fermions, this is not the best adapted representation. The γ's and Ψ's can be rotated with unitary

transformations U, without changing the Lagrangian in Eq. (4.61). Thus a different representation can be obtained by

$$\gamma \to U^\dagger \gamma U . \tag{4.66}$$

This leaves the Clifford algebra unchanged. When transforming the generators $S_{\mu\nu} \to U^\dagger S_{\mu\nu} U$ we have to find a way, to project on the left- and right-handed eigenspaces. For this purpose we define

$$\gamma_5 = \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3 , \tag{4.67}$$

with properties

$$\begin{aligned}
\gamma_5^2 &= \mathbb{1} && \to \text{eigenvalues } \pm 1 \\
\{\gamma_5, \gamma^\mu\} &= 0 \\
\left[S_{\mu\nu}, \gamma_5\right] &= 0 && \to S_{\mu\nu},\ y_5 \text{ can be diagonalised at the same time} .
\end{aligned} \tag{4.68}$$

Note, that in chiral representation $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Consequently, we find the

projection operators on L/R spaces

$$P_{L/R} = \frac{\mathbb{1} \mp \gamma_5}{2} , \tag{4.69}$$

with $P_{L/R}^2 = P_{L/R}$ and $P_L + P_R = \mathbb{1}$. Hence,

$$P_{L/R}\,\Psi = \Psi_{L/R} . \tag{4.70}$$

Next, we discuss **Dirac matrices and Dirac field bilinears**. So far we found that $\bar{\Psi}$ is a Lorentz scalar. One easily finds, that $\bar{\Psi}\gamma^\mu\Psi$ is a 4-vector. Using $\gamma^\mu \to \Lambda^\mu{}_\nu\gamma^\nu$, we find a basis of sixteen 4×4 matrices, defined as antisymmetric combinations of γ-matrices:

$\mathbb{1}$	scalar	1 of these
γ^μ	vector	4 of these
$[\gamma^\mu, \gamma^\nu] := \gamma^{[\mu}\gamma^{\nu]}$	tensor	6 of these
$\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$	pseudo-vector	4 of these
γ_5	pseudo-scalar	1 of these
		16 total

The prefix *pseudo* indicates, that these quantities transform usual under continuous Lorentz transformations, but with an additional sign change under parity transformations. We obtain two conserved currents out of Dirac field bilinears, namely

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi \quad \text{and} \quad j_5^\mu = \bar{\Psi}\gamma^\mu\gamma_5\Psi . \tag{4.71}$$

These are conserved as

$$\begin{aligned}
\partial_\mu \left(\bar{\Psi}\gamma^\mu\Psi\right)\Big|_{\text{EOM}} &= \mathrm{i}\,m\bar{\Psi}\Psi - \mathrm{i}\,m\bar{\Psi}\Psi = 0 \\
\partial_\mu \left(\bar{\Psi}\gamma^\mu\gamma_5\Psi\right) &= 2\mathrm{i}\,m\bar{\Psi}\gamma_5\Psi\Big|_{m=0} = 0 .
\end{aligned} \tag{4.72}$$

Note, that the axial current j_5^μ is only conserved for $m = 0$ (chiral symmetry). The underlying symmetries are

$$\begin{aligned}\Psi \to \mathrm{e}^{\mathrm{i}\alpha}\Psi \quad &\Rightarrow \quad \bar{\Psi} \to \bar{\Psi}\mathrm{e}^{-\mathrm{i}\alpha} \\ \Psi \to \mathrm{e}^{+\mathrm{i}\gamma_5\alpha}\Psi \quad &\Rightarrow \quad \bar{\Psi} \to \bar{\Psi}\mathrm{e}^{+\mathrm{i}\gamma_5\alpha} \,.\end{aligned} \tag{4.73}$$

We will now determine **solutions of the Dirac equation**. As $\Psi(x)$ satisfies the Klein-Gorden equation (2.14), we write

$$\Psi(x) = u(p)\,\mathrm{e}^{-\mathrm{i}px}\,, \tag{4.74}$$

with $u(p)$ being a vector and $p^2 = m^2$. Thus,

$$\mathrm{e}^{-\mathrm{i}px}\,(\mathrm{i}\partial\!\!\!/ - m)\,\Psi(x) = \left(p\!\!\!/ - m\right)u(p) = 0\,. \tag{4.75}$$

Similarly, with $\Psi(x) = v(p)\mathrm{e}^{\mathrm{i}px}$ we find

$$\left(p\!\!\!/ + m\right)v(p) = 0\,, \tag{4.76}$$

for $p^2 = m^2$. We now choose the rest frame as coordinate system. Then:

$$\begin{aligned} p &= (p_0, 0) \\ m\left(\gamma^0 - \mathbb{1}\right)u(p) &= 0\,. \end{aligned} \tag{4.77}$$

With chiral representation, we have $\left(\gamma^0 - \mathbb{1}\right) = \begin{pmatrix} -\mathbb{1}_{2\times2} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & -\mathbb{1}_{2\times2} \end{pmatrix}$. With Dirac representation on the other hand,

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \\ \Rightarrow &\quad \left(\gamma^0 - \mathbb{1}\right) = 2\begin{pmatrix} 0 & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}. \end{aligned} \tag{4.78}$$

With this, we find

$$
\begin{aligned}
u_s(p^0) &= \sqrt{2m}\begin{pmatrix}\chi_s\\0\end{pmatrix}\\
v_s(p^0) &= \sqrt{2m}\begin{pmatrix}0\\ \epsilon\chi_s\end{pmatrix},
\end{aligned}
\tag{4.79}
$$

with $s = 1/2, -1/2$, $\chi_{1/2} = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi_{-1/2} = \begin{pmatrix}0\\1\end{pmatrix}$ and the metric in spinor space $\epsilon = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$. Note, that $\epsilon^{-1}\sigma\epsilon = \bar{\sigma}$. Then we obtain the

general solutions of the Dirac equation

$$
\begin{aligned}
u_s(p) &= \frac{1}{\sqrt{2m}}\frac{\not{p}+m}{\sqrt{p^0+m}}\,u_s(p^0) = \sqrt{p^0+m}\begin{pmatrix}\chi_s\\ \frac{\vec{\sigma}\,\vec{p}}{p^0+m}\chi_s\end{pmatrix}\\
v_s(p) &= \frac{1}{\sqrt{2m}}\frac{\not{p}-m}{\sqrt{p^0+m}}\,v_s(p^0) = -\sqrt{p^0+m}\begin{pmatrix}\frac{\vec{\sigma}\,\vec{p}}{p^0+m}\epsilon\chi_s\\ \epsilon\chi_s\end{pmatrix},
\end{aligned}
\tag{4.80}
$$

where u_s, v_s are normalised to 1. We will now introduce some relations between the solutions. We have

$$
\begin{aligned}
\bar{u}_r(p)\,u_s(p) &= 2m\delta_{rs}\\
\bar{v}_r(p)\,v_s(p) &= -2m\delta_{rs}
\end{aligned}
\tag{4.81}
$$

$$
\bar{u}_r(p)\,v_s(p) = 0 = \bar{v}_r(p)\,u_s(p)
\tag{4.82}
$$

$$
\begin{aligned}
\sum_s u_s(p)_\xi\,\bar{u}_s(p)_{\bar{\xi}} &= (\not{p}+m)_{\xi\bar{\xi}}\\
\sum_s v_s(p)_\xi\,\bar{v}_s(p)_{\bar{\xi}} &= (\not{p}-m)_{\xi\bar{\xi}},.
\end{aligned}
\tag{4.83}
$$

The calculation to Eq. (4.81) goes as follows:

$$
\begin{aligned}
\bar{u}_r(p)\,u_s(p) &= u_r^\dagger(p^0)\,\frac{(\not{p}+m)\,\gamma^0\,(\not{p}+m)}{p^0+m}\,u_s(p^0)\\
\gamma^0\gamma^\dagger\gamma^0=\gamma \quad\rightarrow\quad &= u_r^\dagger(p^0)\,\gamma^0\,\frac{(\not{p}+m)(\not{p}+m)}{p^0+m}\,u_s(p^0)\\
{\scriptstyle u_r^\dagger(p^0)=u_r(p^0),\; u_r(p^0)\gamma^0\gamma^i\,u_s(p^0)=0} \quad\rightarrow\quad &= u_r(p^0)\,\gamma^0\,\frac{p^2+m^2+2p^0m\gamma^0}{p^0+m}\,u_s(p^0)\\
\gamma^0\,u_s(p^0)=u_s(p^0) \quad\rightarrow\quad &= 2u_r(p^0)\,\gamma^0\,\frac{m(p^0+m)}{p^0+m}\,u_s(p^0)\\
&= 2m\delta_{rs}\,,
\end{aligned}
\tag{4.84}
$$

and analogously for $\bar{v}_r\, v_s$. Eq. (4.82) follows from $(\not p - m)(\not p + m) = 0$ and Eq. (4.83) is proven by showing it at the basis $u_s(p)$, $v_s(p)$, i.e.

$$\begin{aligned}
\sum_s u_s(p)\,\bar{u}_s(p)\,u_r(p) &= \sum_s u_s(p)\,2m\delta_{rs} \\
&= 2m\,u_r(p) = \frac{2m(\not p + m)}{\sqrt{p^0 + m}}\,u_r(p^0) = \frac{(\not p + m)^2}{\sqrt{p^0 + m}}\,u_r(p^0) \\
&= (\not p + m)\,u_r(p) \qquad (4.85) \\
\sum_s u_s(p)\,\bar{u}_s(p)\,v_r(p) &= 0 = (\not p + m)\,v_r(p)\,, \qquad (4.86)
\end{aligned}$$

and similarly for $\sum_s v_s(p)\,\bar{v}_s(p)$.

III. Quantisation

First, we try to quantise fermions similarly to scalars (bosons), as performed in section III. In analogy to Eq. (2.126), we have the

general solution to the Dirac equation

$$\Psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{1}{\sqrt{2p^0}}\,\sum_s \Big[\mathrm{e}^{-\mathrm{i}px}\,a_s(\mathbf{p})\,u_s(\mathbf{p}) + \mathrm{e}^{+\mathrm{i}px}\,b_s^\dagger(\mathbf{p})\,v_s(\mathbf{p})\Big]\,, \quad \text{with} \quad p^0 = \sqrt{\mathbf{p}^2 + m^2}\,. \qquad (4.87)$$

The Hamiltonian follows from Eq. (4.65)

$$\begin{aligned}
H &= \int \mathrm{d}^3 x\,\mathcal{H} = \int \mathrm{d}^3 x\,\Psi^\dagger(\mathbf{x})\,\gamma^0\,\left(\mathrm{i}\,\vec{\gamma}\,\vec{\partial} + m\right)\,\Psi(\mathbf{x}) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{2p^0}{2p^0}\,p^0\,\sum_s \Big[a_s^\dagger(\mathbf{p})\,a_s(\mathbf{p}) - b_s(\mathbf{p})\,b_s^\dagger(\mathbf{p})\Big]\,. \qquad (4.88)
\end{aligned}$$

Note the "−" sign instead of the "+" in Eq. (2.129) in the last line! Here, we have used Eq. (4.81) and

$$\begin{aligned}
(\vec{\gamma}\,\mathbf{p} + m)\;u(p) &= \Big(-(\not p - m) + \gamma^0\,p^0\Big)u(p) \\
&= \gamma^0\,p^0\,u(p) \\
(-\vec{\gamma}\,\mathbf{p} + m)\;v(p) &= -\gamma^0\,p^0\,v(p)\,, \qquad (4.89)
\end{aligned}$$

where the "−" in the last line corresponds to the "−" in Eq. (4.88). If we now suggest commuting operators, e.g.

$$b_s\,b_s^\dagger = b_s^\dagger\,b_s + \text{c-number}\,,$$

this would imply that

$$H \simeq \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{2p^0}{2p^0}\,p^0\,\sum_s \Big[a_s^\dagger(\mathbf{p})\,a_s(\mathbf{p}) - b_s^\dagger(\mathbf{p})\,b_s(\mathbf{p})\Big]\,,$$

which does not hold, due to the minus sign. Therefore, we suggest

$$b_s\, b_s^\dagger = -b_s^\dagger\, b_s + \text{c-number}\,. \tag{4.90}$$

Further, demanding

$$\left[\Psi(\mathbf{x}), \mathrm{i}\,\Psi^\dagger(\mathbf{y})\right] = \mathrm{i}\,\delta(\mathbf{x}-\mathbf{y})\,, \tag{4.91}$$

implies

$$\begin{aligned}\left[a_s(\mathbf{p}), a_r^\dagger(\mathbf{q})\right] &= (2\pi)^3\,\delta(\mathbf{p}-\mathbf{q})\,\delta_{sr} \\ &= -\left[b_s(\mathbf{p}), b_r^\dagger(\mathbf{q})\right]\,.\end{aligned} \tag{4.92}$$

Again, note the additional "−" int the last line, which rescues causality, but does not cure the issue with the minus sign in the Hamiltonian! Hence, we define the

anti-commutation relations of creation and annihilation operator (for fermions)

$$\begin{aligned}\left\{a_s(\mathbf{p}), a_r^\dagger(\mathbf{q})\right\} &= (2\pi)^3\,\delta_{sr}\,\delta(\mathbf{p}-\mathbf{q}) \\ \left\{b_s(\mathbf{p}), b_r^\dagger(\mathbf{q})\right\} &= (2\pi)^3\,\delta_{sr}\,\delta(\mathbf{p}-\mathbf{q})\,.\end{aligned} \tag{4.93}$$

Note, that the anti-commutators of a-a, b-b, b-$a^{(\dagger)}$ vanish and in particular $a_s(\mathbf{p})\,a_s(\mathbf{p}) = a_s^2 = 0$ (Grassmann variables). It follow the

anti-commutation relations of field operators (for fermions)

$$\begin{aligned}\left\{\Psi_\xi(\mathbf{x}), \Psi_{\xi'}^\dagger(\mathbf{y})\right\} &= \delta_{\xi\xi'}\,\delta(\mathbf{x}-\mathbf{y}) \\ \left\{\Psi_\xi(\mathbf{x}), \Psi_\xi(\mathbf{y})\right\} = 0 &= \left\{\Psi_\xi^\dagger(\mathbf{x}), \Psi_\xi^\dagger(\mathbf{y})\right\}\,.\end{aligned} \tag{4.94}$$

Similarly to section III, we will now construct the **Fock space**. Again, we define a vacuum state $|0\rangle$ (compare to Eq. (2.90)), with

$$\sqrt{2w_\mathbf{p}}\,a_s(\mathbf{p})\,|0\rangle = 0 = \sqrt{2w_\mathbf{p}}\,b_s(\mathbf{p})\,|0\rangle\ . \tag{4.95}$$

The one-particle states are given by

$$|\mathbf{p}, s\rangle = \sqrt{2w_\mathbf{p}}\,a_s^\dagger(\mathbf{p})\,|0\rangle\ , \tag{4.96}$$

and $\sqrt{2w_\mathbf{p}}\,b_s^\dagger(\mathbf{p})\,|0\rangle$ for anti-particles. The states are normalised to

$$\langle\mathbf{q}, r|\mathbf{p}, s\rangle = (2\pi)^3\,2p^0\,\delta_{rs}\,\delta(\mathbf{p}-\mathbf{q})\,. \tag{4.97}$$

Note, that the states are *antisymmetric*, as e.g. for two-particle states:

$$\sim a_s^\dagger(\mathbf{p})\,a_r^\dagger(\mathbf{q})\,|0\rangle = -\,a_r^\dagger(\mathbf{q})\,a_s^\dagger(\mathbf{p})\,|0\rangle\ . \tag{4.98}$$

In particular, it is

$$a_r^\dagger(\mathbf{p})\,a_r^\dagger(\mathbf{p})\,|0\rangle = 0\,, \tag{4.99}$$

which mirrors the Fermi-exclusion principle. Next, we consider **continuous symmetries**. Again, we define the 4-momentum operator (compare to Eq. (2.120)):

$$\begin{aligned} P^0 &= \int \frac{d^3p}{(2\pi)^3} p^0 \sum_s \left[a_s^\dagger(\mathbf{p})\, a_s(\mathbf{p}) + b_s^\dagger(\mathbf{p})\, b_s(\mathbf{p})\right], \quad p^0 = E > 0 \\ &= H \\ P^i &= \int \frac{d^3p}{(2\pi)^3} p^i \sum_s \left[a_s^\dagger(\mathbf{p})\, a_s(\mathbf{p}) + b_s^\dagger(\mathbf{p})\, b_s(\mathbf{p})\right]. \end{aligned} \tag{4.100}$$

Ψ is a complex field and the Lagrangian is invariant under $\Psi \to e^{ie\alpha}\Psi$, $\bar{\Psi} \to \bar{\Psi}\, e^{-ie\alpha}$, as shown in Eq. (4.73). This leads to conserved currents and similarly to Eq. (2.61), we can formulate the

Noether charge (for fermions)

$$\begin{aligned} Q &= \int d^3x\, j^0 = e\int d^3x\, \Psi^\dagger(x)\, \Psi(x) = e\int d^3x\, \bar{\Psi}(x)\, \gamma^0\, \Psi(x) \\ &\simeq e\int \frac{d^3p}{(2\pi)^3} \sum_s \left[a_s^\dagger(\mathbf{p})\, a_s(\mathbf{p}) - b_s^\dagger(\mathbf{p})\, b_s(\mathbf{p})\right], \end{aligned} \tag{4.101}$$

where e is the elementary charge, and $\left(a_s^\dagger(\mathbf{p})\, a_s(\mathbf{p})\right)$, $\left(b_s^\dagger(\mathbf{p})\, b_s(\mathbf{p})\right)$ correspond to a fermion with charge e and an anti-fermion with charge $-e$, respectively. Let us now calculate the **propagator**. Therefore we first consider

$$\begin{aligned} \langle 0|\, \Psi_\xi(x)\, \bar{\Psi}_{\xi'}(y)\, |0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left[\sum_s (u_s)_\xi\, (\bar{u}_s)_{\xi'}\right] e^{-ip(x-y)} \\ &= (i\not{\partial}_x + m)_{\xi\xi'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\, e^{-ip(x-y)} \end{aligned} \tag{4.102}$$

and

$$\langle 0|\, \bar{\Psi}_{\xi'}(y)\, \Psi_\xi(x)\, |0\rangle = -(i\not{\partial}_x + m)_{\xi\xi'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\, e^{-ip(y-x)}. \tag{4.103}$$

Note, that the two rear integrals correspond to the scalar propagator (see Eq. (3.85)) without the θ-function from time ordering. Also note the global minus sign in Eq. (4.103), which implies that the order of $\Psi(x)$, $\bar{\Psi}(y)$ is important (which was not the case for the scalar field)! With time ordering we find in analogy to Eq. (3.87) the

Feynman-propagator (for fermions)

$$\begin{aligned} S_F(x-y) &= \langle 0|\, T\ \Psi_\xi(x)\, \bar{\Psi}_{\xi'}(y)\, |0\rangle \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i\,(\not{p} + m)_{\xi\xi'}}{p^2 - m^2 + i\epsilon}\, e^{-ip(x-y)}, \end{aligned} \tag{4.104}$$

with time ordering

$$\begin{aligned} T\ \Psi(x)\bar{\Psi}(y) &= \theta(x^0 - y^0)\, \Psi(x)\, \bar{\Psi}(y) - \theta(y^0 - x^0)\, \bar{\Psi}(y)\, \Psi(x) \\ &= -T\ \bar{\Psi}(y)\, \Psi(x) \end{aligned} \tag{4.105}$$

Again, note the relative minus sign, when comparing to Eq. (3.35).
We can now formulate the **Feynman rules for fermions**. We can directly take over the results for the scalar theory (section III- V), but we have to take care of the anti-symmetry of fermions. We have already introduced

$$T\,\Psi\bar{\Psi} = -T\,\bar{\Psi}\Psi\,.$$

Accordingly, if we define contractions as in the scalar theory, it follows

$$\begin{aligned} \overbrace{\Psi(x)\,\bar{\Psi}(y)} &= \langle 0|\,T\,\Psi(x)\,\bar{\Psi}(y)\,|0\rangle \\ &= S_F(x-y) \\ &= -\overbrace{\bar{\Psi}(y)\,\Psi(x)}\,. \end{aligned} \tag{4.106}$$

Then

$$\cdots\overbrace{\Psi\,\Psi^n\,\bar{\Psi}^m\,\bar{\Psi}}\cdots = (-1)^{n+m}\cdots\overbrace{\Psi\,\bar{\Psi}}\,\Psi^n\,\bar{\Psi}^m\,. \tag{4.107}$$

Also it holds for normal ordering:

$$\begin{aligned} :a\,a^\dagger: &= -:a^\dagger\,a: = -a^\dagger\,a \\ \Rightarrow\quad :\Psi_1\cdots\Psi_n\Psi_{n+1}\cdots: &= -:\Psi_1\cdots\Psi_{n+1}\Psi_n\cdots: \\ \Rightarrow\quad :\Psi_1\cdots\Psi_n\bar{\Psi}_{n+1}\cdots: &= -:\Psi_1\cdots\bar{\Psi}_{n+1}\Psi_n\cdots: \end{aligned} \tag{4.108}$$

Similarly to Eq. (3.93), we obtain

Wick's theorem (for fermions)

$$T\,\Psi(x_1)\cdots\Psi(x_n)\,\bar{\Psi}(x_{n+1})\cdots\bar{\Psi}(x_{n+m}) = :\Psi(x_1)\cdots\Psi(x_n)\,\bar{\Psi}(x_{n+1})\cdots\bar{\Psi}(x_{n+m}) + \text{all contractions}:\,. \tag{4.109}$$

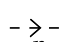
In the following, let us discuss the simplest **interacting theory** with fermions. Then we have a Lagrangian of the form

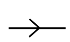
$$\mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{I}}\,, \tag{4.110}$$

with the interaction Lagrangian of *Yukawa theory*

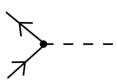
$$\mathcal{L}_{\text{I}} = -h\,\bar{\Psi}\,\phi\,\Psi\,, \tag{4.111}$$

where h is the Yukawa coupling. We obtain
Propagators:

$\phi:\quad \overbrace{\phi\,\phi} =$  p $= \dfrac{\mathrm{i}}{p^2 - m_\phi^2 + \mathrm{i}\epsilon}$

$\Psi:\quad \overbrace{\Psi\,\bar{\Psi}} =$  p $= \dfrac{\mathrm{i}(\not{p} + m_\Psi)}{p^2 - m_\Psi^2 + \mathrm{i}\epsilon}$

Vertex:

 $= -\mathrm{i}\,h$

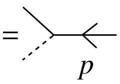
(4.112)

External leg contraction:

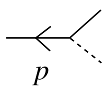
$$\phi\,|\mathbf{p}\rangle \quad := 1 =: \quad \langle \mathbf{p}|\,\phi$$

$$\begin{aligned}
\Psi(x)\Big|_{\text{annihilation}}|\mathbf{p},s\rangle &= \int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{2p^0}{2q^0}}\sum_r\left[\mathrm{e}^{-\mathrm{i}qx}\,u_r(\mathbf{q})\,a_r(\mathbf{q})\,a_s^\dagger(\mathbf{p})\,|0\rangle\right]\\
&= \int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{2p^0}{2q^0}}\sum_r\left[\mathrm{e}^{-\mathrm{i}qx}\,u_r(\mathbf{q})\,\left\{a_r(\mathbf{q}),a_s^\dagger(\mathbf{p})\right\}|0\rangle\right]\\
&= \mathrm{e}^{-\mathrm{i}px}\,u_s(p)\,. \qquad (4.113)
\end{aligned}$$

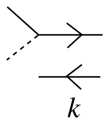
We drop the phase in Eq. (4.113) and find:

$$\Psi\,|\mathbf{p},s\rangle := u_s(\mathbf{p}) =$$ 

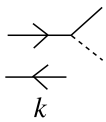
Analogously:

$$\langle \mathbf{p},s|\,\bar\Psi := \bar u_s(\mathbf{p}) =$$ 

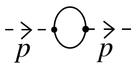
Anti-fermions:



$$= \bar\Psi\,|\mathbf{k},s\rangle := \bar v^s(\mathbf{k})$$
$$\shortparallel$$
$$b_s^\dagger(\mathbf{k})\,|0\rangle$$



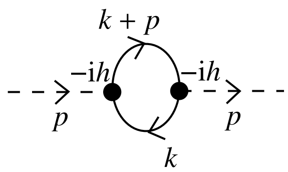
$$= \langle \mathbf{k},s|\,\Psi := v^s(\mathbf{k})$$

Loops: e.g. 

(vacuum polarisation)

$$\langle \mathbf{q}|\,(\phi\,\bar\Psi\,\Psi)\,(\phi\,\bar\Psi\,\Psi)\,|\mathbf{p}\rangle \sim -\,\langle \mathbf{q}|\,\phi\,(\Psi\,\bar\Psi\,\Psi\,\bar\Psi)\,\phi\,|\mathbf{p}\rangle\;. \qquad (4.114)$$

Consequently, closed fermionic loops lead to minus signs! The calculations for loop integrals and Dirac traces goes as follows. Consider



.

Then:

$$-(-\mathrm{i}\,h)^2\int \mathrm{d}^4x\int \mathrm{d}^4y\,\langle \mathbf{q}|\,\phi_x\,\Psi_{x_\xi}\,\bar\Psi_{y_\eta}\,\Psi_{y_\eta}\,\bar\Psi_{x_\xi}\,\phi_y\,|\mathbf{p}\rangle\;. \qquad (4.115)$$

We now use, that for any (is this correct?) operators O_i it holds:

$$\int \mathrm{d}^4x\int \mathrm{d}^4y\,O_1(x,y)\,O_2(y,x') = \int \mathrm{d}^4x\,O_3(x,x')\,, \qquad (4.116)$$

which gives the trace. Thus

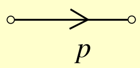
$$
\begin{aligned}
\overbracket{\Psi_{x_\xi}\,\bar{\Psi}_{y_\eta}}\,\overbracket{\Psi_{y_\eta}\,\bar{\Psi}_{x_\xi}} &= S_{F_{\xi\eta}}(x-y)\,S_{F_{\eta\xi}}(y-x) \\
&= \mathrm{tr}_{\text{Dirac}}\left[S_F(x-y)\,S_F(y-x)\right]
\end{aligned}
\tag{4.117}
$$

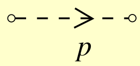
It follows

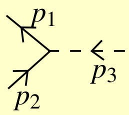
$$
\begin{aligned}
&-(-\mathrm{i}\,h)^2\int \mathrm{d}^4x\int \mathrm{d}^4y\,\langle\mathbf{q}|\,\phi_x\,\overbracket{\Psi_{x_\xi}\,\bar{\Psi}_{y_\eta}}\,\overbracket{\Psi_{y_\eta}\,\bar{\Psi}_{x_\xi}}\,\phi_y\,|\mathbf{p}\rangle \\
&\simeq -h^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{tr}\left(\frac{\not{k}+m_\Psi}{k^2-m_\Psi^2+\mathrm{i}\epsilon}\,\frac{\slashed{(k+p)}+m_\Psi}{(k+p)^2-m_\Psi^2+\mathrm{i}\epsilon}\right)
\end{aligned}
\tag{4.118}
$$

Let us summarise the

Feynman rules (in momentum space, for Yukawa theory)

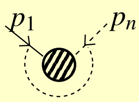
i)  $= \dfrac{\mathrm{i}(\not{p}+m_\Psi)}{p^2-m_\Psi^2+\mathrm{i}\epsilon}$

 $= \dfrac{\mathrm{i}}{p^2-m_\phi^2+\mathrm{i}\epsilon}$

ii)  $= -\mathrm{i}\,h$ and $p_2 = -(p_1+p_3)$ (momentum conservation)

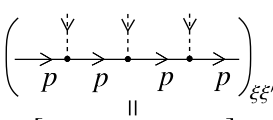
iii) $\displaystyle\int\frac{\mathrm{d}^4}{(2\pi)^4}$ for each loop

$(-)$ for each fermion loop

iv) $(2\pi)^4\,\delta^4(\sum_i p_i)$ for  .

(4.119)

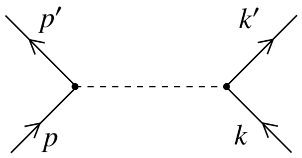
When comparing to Eq. (3.115), we note, that there is *no symmetry factor* in Eq. (4.119), as $\mathcal{L}_I$ is built-up from 3 different fields. Also, now the *direction of the fermion line is important*. Along fermion lines Dirac indices are contracted, e.g.



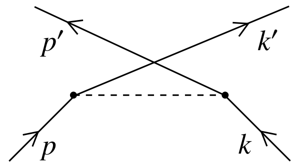
$$
\simeq (-\mathrm{i}\,h)^3\left(\frac{\mathrm{i}\,(\not{p}+m_\Psi)}{p^2-m_\Psi^2+\mathrm{i}\epsilon}\right)_{\xi\xi'}(-\mathrm{i}\,h)\left(\frac{\mathrm{i}\,(\not{p}+m_\Psi)}{p^2-m_\Psi^2+\mathrm{i}\epsilon}\right)_{\eta\xi'}.
$$

$$
\bar{\Psi}\,\phi\left[\left(\Psi\,\bar{\Psi}\right)\phi\left(\Psi\,\bar{\Psi}\right)\right]_{\xi\xi'}\phi\,\Psi
$$

Example 13: scattering process.



$+$



$$\Rightarrow \mathrm{i}\, M = (-\mathrm{i}\, h)^2 \left[\bar{u}(\mathbf{p}')\, u(\mathbf{p})\, \frac{1}{(p-p')^2 - m_\phi^2}\, \bar{u}(\mathbf{k}')\, u(\mathbf{k}) - \bar{u}(\mathbf{p}')\, u(\mathbf{k})\, \frac{1}{(p-k)^2 - m_\phi^2}\, \bar{u}(\mathbf{k}')\, u(\mathbf{p}) \right] . \quad (4.120)$$

Example 14: QED: couple electron Ψ_e to photon A_μ.

The interaction Lagrangian is then contracted with a vector A_μ

$$\mathcal{L}_I = e\, \bar{\Psi}\, A_\mu\, \gamma^\mu\, \Psi\, . \quad (4.121)$$

Then

$$\mathcal{L}_{\mathrm{QED}} = \mathcal{L}_{\mathrm{photon}} + \mathcal{L}_{\mathrm{Dirac}} + \mathcal{L}_{\mathrm{I}}\, , \quad (4.122)$$

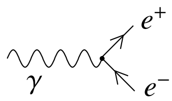
with

$$\mathcal{L}_{\mathrm{Dirac}} + \mathcal{L}_{\mathrm{I}} = \bar{\Psi}\, (\mathrm{i}\, \not{D} - m)\, \Psi\, , \quad (4.123)$$

where

$$D_\mu = \partial_\mu - \mathrm{i}\, e\, A_\mu \quad (4.124)$$

For the vertices it is



$= \mathrm{i}\, e$

To explicitly compute expressions as the above, we need the photon propagator. This will be subject to the subsequent chapter, in particular section II.

5. Gauge Fields

I. Gauge Symmetry

Consider the Dirac theory of e^+, e^-

$$\mathcal{L}_D = \bar{\Psi}(x)\ (\mathrm{i}\,\partial\!\!\!/ - m)\ \Psi(x)\,, \tag{5.1}$$

or complex scalar theory

$$\mathcal{L}_\phi = \partial_\mu\phi\,\partial_\mu\phi^* - m^2\,\phi\phi^* - V\,(\phi\phi^*)\ . \tag{5.2}$$

The Lagrangians in Eq. (5.1) and Eq. (5.2) are invariant under global U(1)-rotations, namely

$$\begin{aligned} \Psi &\rightarrow \mathrm{e}^{\mathrm{i}\alpha}\,\Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi}\,\mathrm{e}^{-\mathrm{i}\alpha} \\ \phi &\rightarrow \mathrm{e}^{\mathrm{i}\alpha}\,\phi, \quad \phi^* \rightarrow \phi^*\,\mathrm{e}^{-\mathrm{i}\alpha}\,, \end{aligned} \tag{5.3}$$

which corresponds to a global rotation in field space. Let us require the invariance of the theory under **local rotations** (*gauge symmetry*), e.g.

$$\Psi(x) \rightarrow \mathrm{e}^{\mathrm{i}\alpha(x)}\,\Psi(x)\,. \tag{5.4}$$

We see, that $\mathcal{L}_D$ is not invariant, as

$$\begin{aligned} \mathcal{L}_D &\rightarrow \mathcal{L}_D - \bar{\Psi}(\partial\!\!\!/\,\alpha)\Psi \\ &= \mathcal{L}_D - \partial_\mu\alpha\, j^\mu\,, \end{aligned} \tag{5.5}$$

with

$$j^\mu = \bar{\Psi}\,\gamma^\mu\,\Psi\,. \tag{5.6}$$

Hence, if we add a term $A_\mu j^\mu$ to $\mathcal{L}_D$, and demand invariance, it follows

$$\begin{aligned} \mathcal{L}_D + A_\mu j^\mu &\rightarrow \mathcal{L}_D - \partial_\mu\alpha\, j^\mu + A'_\mu j^\mu \\ &\overset{!}{=} \mathcal{L}_D + A_\mu j^\mu \\ \Rightarrow A_\mu(x) &\rightarrow A_\mu(x) + \partial_\mu\alpha(x)\,. \end{aligned} \tag{5.7}$$

Also note, that $\mathcal{L}$ is a Lorentz scalar:

$$A_\mu \overset{\Lambda}{\longrightarrow} \Lambda^\mu{}_\nu A_\nu\,, \tag{5.8}$$

as $A_\mu j^\mu$ transforms as a scalar. Next, we write the invariant action

$$\mathcal{L}_D = \bar{\Psi}\ (\mathrm{i}\,D\!\!\!\!/ - m)\ \Psi\,, \tag{5.9}$$

with the *covariant* derivative

$$D_\mu = \partial_\mu - \mathrm{i}A_\mu\,. \tag{5.10}$$

A_μ is also called a connection (German: "Zusammenhang"). It induces covariant transformation properties for D_μ:

$$D_\mu \quad \to \quad e^{i\alpha(x)}\, D_\mu\, e^{-i\alpha(x)} = \partial_\mu - iA_\mu - i\partial_\mu\alpha = \partial_\mu - iA'_\mu$$

$$\Rightarrow D_\mu\Psi \quad \to \quad e^{i\alpha(x)}\, D_\mu\, e^{-i\alpha(x)}\, e^{i\alpha(x)}\, \Psi = e^{i\alpha(x)}\, D_\mu\Psi \quad \text{(transforms homog. as the field } \Psi\text{)}\,, \tag{5.11}$$

as well as $D_\mu\phi \to e^{i\alpha(x)}\, D_\mu\phi$. Similarly, we get that

$$\mathcal{L}_\phi = D_\mu\phi\, \left(D_\mu\phi\right)^* - m^2\, \phi\phi^* - V\left(\phi\phi^*\right) \tag{5.12}$$

is invariant under

$$\begin{aligned} \phi(x) \quad &\to \quad e^{i\alpha(x)}\, \phi(x) \\ A_\mu \quad &\to \quad A_\mu + \partial_\mu\alpha\,. \end{aligned} \tag{5.13}$$

To examine the dynamics of the gauge field A_μ, we start by constructing gauge-invariant scalar quantities from A_μ. This is easily done from D_μ, which transforms covariantly:

$$\left[D_\mu, D_\nu\right] = -i\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = -i\, F_{\mu\nu}\,, \tag{5.14}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, or $F_{\mu\nu} = i\left[D_\mu, D_\nu\right]$. As shown in Eq. (5.8), A_μ transforms as a vector. Thus

$$F_{\mu\nu} \quad \to \quad \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma} F_{\rho\sigma} \tag{5.15}$$

transforms as tensor. $F_{\mu\nu}$ can be interpreted as field strength, or curvature. $F_{\mu\nu}$ is gauge invariant:

$$\begin{aligned} F_{\mu\nu} \quad &\to \quad i\left[e^{-i\alpha}\, D_\mu\, e^{i\alpha}, e^{-i\alpha}\, D_\nu\, e^{i\alpha}\right] \\ &= \quad i\, e^{-i\alpha}\left[D_\mu, D_\nu\right] e^{i\alpha} = i\, e^{-i\alpha}\left(-i\, F_{\mu\nu}\right) e^{i\alpha} \\ &= \quad F_{\mu\nu}\,. \end{aligned} \tag{5.16}$$

In summary this means, that $F_{\mu\nu}$ is gauge invariant, but is a Lorentz tensor. Thus, $F_{\mu\nu}F^{\mu\nu}$ is gauge invariant and a Lorentz scalar. Therefore, a gauge invariant Lagrangian can be written as

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4e^2}\, F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_D\,. \tag{5.17}$$

We re-parametrise $A_\mu \to eA_\mu$, with the electric charge e and obtain

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\left(i\, \not{D} - m\right)\Psi\,, \tag{5.18}$$

with

$$D_\mu = \partial_\mu - i\, eA_\mu\,. \tag{5.19}$$

Note, that this construction also goes through for

$$\begin{aligned} \Psi \quad &\to \quad U\,\Psi, \quad \text{for} \quad U \in \text{SU(N)} \\ D_\mu \quad &\to \quad U\, D_\mu\, U^{-1} \\ F_{\mu\nu}F^{\mu\nu} \quad &\to \quad \sum_{a=1}^{N^2-1}\left(F_{\mu\nu}\right)^a\left(F^{\mu\nu}\right)^a\,, \end{aligned} \tag{5.20}$$

with

$$F_{\mu\nu} = \frac{i}{e}\left[D_\mu, D_\nu\right] \sim \left[A_\mu, A_\nu\right]\,. \tag{5.21}$$

We remark, that we have quantum chromodynamics for $N = 3$ and weak interaction for $N = 2$.

II. Quantisation

To quantise gauge fields, we concentrate on the pure gauge field Lagrangian

$$\mathcal{L}_A = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{5.22}$$

The equation of motion is

$$\partial_\mu \frac{\partial \mathcal{L}_\mu}{\partial_\mu A_\mu} = \partial_\mu F^{\mu\nu} = \left(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma\right) A_\sigma = 0\,, \tag{5.23}$$

with the current $\partial_\mu F^{\mu\nu} = J^\nu$. Eq. (5.23) reflects a redundancy of the gauge field A_μ, because the EOM is invariant under

$$\begin{aligned} A_\mu &\to A_\mu + e\partial_\mu \alpha \\ \partial_\mu F^{\mu\nu} + e\left(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma\right)\partial_\sigma \alpha &= \partial_\mu F^{\mu\nu} + 0\,, \end{aligned} \tag{5.24}$$

since

$$\begin{aligned} \partial_\mu \partial^\mu \eta^{\nu\sigma}\partial_\sigma \alpha - \partial^\nu \partial^\sigma \partial_\sigma \alpha &= \partial^2 \partial^\nu \alpha - \partial^2 \partial^\nu \alpha = 0 \\ \left(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma\right)\partial_\sigma &\mathrel{\hat{=}} 0\,. \end{aligned} \tag{5.25}$$

In momentum space this writes

$$\left(p^2 \eta^{\nu\sigma} - p^\nu p^\sigma\right) p_\sigma \mathrel{\hat{=}} 0\,, \tag{5.26}$$

where the term in the brackets is the transverse part, which we will discuss later in the section. Eq. (5.23) and Eq. (5.24) already entail, that A^μ cannot have canonical commutation relations! But what about the canonical momentum Π^μ:

$$\begin{aligned} \Pi^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -\frac{1}{4}\frac{\partial}{\partial(\partial_0 A_\mu)}\left(F_{\rho\sigma}F_{\gamma\delta}\eta^{\sigma\delta}\eta^{\rho\gamma}\right) \\ &= -\frac{1}{2}F_{\rho\sigma}\eta^{\sigma\delta}\eta^{\rho\gamma}\frac{\partial F_{\gamma\delta}}{\partial(\partial_0 A_\mu)} \\ &= F^{\mu 0}\,. \end{aligned} \tag{5.27}$$

In particular, it is

$$\Pi^0 \mathrel{\hat{=}} 0\,, \tag{5.28}$$

which also reflects the redundancy. We remove the redundancy by **fixing the gauge**, e.g. with a Lorentz- or covariant gauge

$$\partial_\mu A^\mu = 0\,. \tag{5.29}$$

For these A^μ we can write

$$\begin{aligned} \mathcal{L}_A &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\epsilon}\left(\partial_\mu A^\mu\right)^2 \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \end{aligned} \tag{5.30}$$

or

$$S[A] = \frac{1}{2}\int \mathrm{d}^4x\, A_\mu \left(\partial_\rho\partial^\rho\eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right) A_\nu\,. \tag{5.31}$$

We split the gauge field in transverse and longitudinal parts

$$A_\mu = (A_\perp)_\mu + (A_L)_\mu\ , \tag{5.32}$$

with

$$\begin{aligned} \partial_\mu\,(A_\perp)_\mu &= 0 \\ \left(\partial_\mu\partial^\mu\eta^{\nu\sigma} - \partial^\nu\partial^\sigma\right)(A_L)_\nu &= 0\,. \end{aligned} \tag{5.33}$$

It follows, that

$$\begin{aligned} \left(\partial_\mu\partial^\mu\eta^{\nu\sigma} - \partial^\nu\partial^\sigma\right)(A_\perp)_\nu &= \frac{1}{\xi}\partial^\sigma\partial_\nu A_L^\nu \\ &= 0\,, \end{aligned} \tag{5.34}$$

because the left-hand side is solely transverse, and the right-hand side solely longitudinal. The EOM is given by

$$\partial_\mu F^{\mu\nu} = -\frac{1}{\xi}\partial^\nu\left(\partial_\mu A^\mu\right) = 0\,. \tag{5.35}$$

Note, that $\left(\partial_\rho\partial^\rho\eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\nu\partial^\sigma\right)$ is invertible, and specifically simple for $\xi = 1$ (Feynman gauge $\partial_\mu\partial^\mu\eta^{\nu\sigma}$). With Eq. (5.29) we obtain the

EOM for the Lorentz gauge

$$\partial_\rho\partial^\rho A^\nu = 0\,, \tag{5.36}$$

which is similar to the Klein-Gordon equation (Eq. (2.14)). Eq. (5.36) suggests a quantised field

$$A_\mu(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\left(\mathrm{e}^{-\mathrm{i}kx}\, a_\mu(\mathbf{k}) + \mathrm{e}^{\mathrm{i}kx}\, a_\mu^\dagger(\mathbf{k})\right)\,, \tag{5.37}$$

with the commutation relations

$$\begin{aligned} \left[a_\mu(\mathbf{k}), a_\nu^\dagger(\mathbf{k}')\right] &= -\eta_{\mu\nu}\,(2\pi)^3\delta(\mathbf{k} - \mathbf{k}') \\ \left[a_\mu(\mathbf{k}), a_\nu(\mathbf{k}')\right] &= 0 = \left[a_\mu^\dagger(\mathbf{k}), a_\nu^\dagger(\mathbf{k}')\right]\,. \end{aligned} \tag{5.38}$$

Note, that the η in the first equation is necessary for Lorentz-symmetry. However, Eq. (5.37) and Eq. (5.38) are not compatible with Eq. (5.29), as

$$\begin{aligned} \partial_\mu A^\mu &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{-\mathrm{i}}{\sqrt{2k^0}}\left(\mathrm{e}^{-\mathrm{i}kx}\, k_\mu\, a^\mu(\mathbf{k}) + \mathrm{e}^{\mathrm{i}kx}\, k_\mu\left(a^\dagger\right)^\mu(\mathbf{k})\right) \\ &\overset{!}{=} 0\,. \end{aligned} \tag{5.39}$$

This entails, that $k_\mu a^\mu(\mathbf{k}) \stackrel{!}{=} 0$, because if Eq. (5.39) fails, the EOM is *not* satisfied

$$\partial_\mu F^{\mu\nu} = -\partial^\nu \partial_\mu A^\nu \,. \tag{5.40}$$

However,

$$\begin{aligned} k^\mu \left[a_\mu(\mathbf{k}), a_\nu^\dagger(\mathbf{k}')\right] &= -k^\nu \, (2\pi)^3 \delta(\mathbf{k}-\mathbf{k}') \\ &\neq 0 \,. \end{aligned} \tag{5.41}$$

Indeed one can show, that it is *not* possible to quantise the gauge field A_μ with canonical commutation relations *and* $\partial_\mu A^\mu = 0$, or other gauge conditions; If using A^μ in Eq. (5.37) and Eq. (5.38), the gauge $\partial_\mu A^\mu$ has to be implemented on the states! We will target this problem later in this section, but prior to this, we construct the **Fock space** $\mathcal{F}$ based on Eq. (5.37) and Eq. (5.38). We define the vacuum state $|0\rangle$ with

$$\langle 0|0\rangle = 1 \,. \tag{5.42}$$

One-particle states are given by

$$\sqrt{2k^0}\, a_\mu^\dagger(\mathbf{k})\, |0\rangle \,, \tag{5.43}$$

with norm

$$\sqrt{2k^0 2\,(k')^0}\; \langle 0|\, a_\nu(\mathbf{k}')\, a_\mu^\dagger(\mathbf{k})\, |0\rangle = -\eta_{\mu\nu}\, (2\pi)^3\, 2k^0\, \delta(\mathbf{k}-\mathbf{k}') \,. \tag{5.44}$$

Thus, we have positive norm states for $\mu = \nu = i$, and negative norm states for $\mu = \nu = 0$. Consequently, $\mathcal{F}$ is not the physical Hilbert space $\mathcal{H}$, as it does not allow for probability interpretation. We remark, that $\eta_{\mu\nu} \to \eta^{\mu\nu}$ does not solve the problem of negative norm states (leave aside the wrong commutators $\left[A^i, \Pi^i\right]$). But separating the positive norm subspace of $\mathcal{F}$, will solve all problems of quantisation. This is the **Gupta-Bleuler quantisation**. We demand, that the EOM is satisfied on

physical states

$$\left\langle \text{physical states}' \right| \partial_\mu F^{\mu\nu} \left| \text{physical states} \right\rangle \stackrel{!}{=} 0 \,, \tag{5.45}$$

that is, its matrix elements vanish. Eq. (5.45) is satisfied for

$$k^\mu a_\mu(\mathbf{k}) \left|\text{physical states}\right\rangle = 0 \,, \tag{5.46}$$

which is trivially satisfied on the vacuum. The above suggests to rewrite A_μ in Eq. (5.37) as

field operator (Lorentz gauge)

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \sum_{\lambda=0}^{3} \left(\alpha_\lambda(\mathbf{k})\, \epsilon_\mu^\lambda(k)\, e^{-ikx} + \alpha_\lambda^\dagger(\mathbf{k}) \left(\epsilon_\mu^\lambda\right)^* (k)\, e^{ikx} \right), \tag{5.47}$$

where the ϵ_μ^λ introduce unitary rotations[1] from a_μ to a_λ with

$$\begin{aligned} \epsilon_\mu^\lambda(k)\, \epsilon^{\lambda'\mu^*}(k) &= \eta^{\lambda\lambda'} \\ \epsilon_\mu^\lambda(k)\, \epsilon_{\lambda\nu}^*(k) &= \eta_{\mu\nu}\,. \end{aligned} \tag{5.48}$$

Hence, we write the "new" operators α as linear combination of the "old" operators a:

$$\alpha_\lambda(\mathbf{k}) = a_\mu(\mathbf{k})\epsilon_\lambda^\mu(k)\,. \tag{5.49}$$

Now, we choose our coordinate system without loss of generality, such that $k{\cdot}\epsilon^0 = k^0 = k{\cdot}\epsilon^3$ and $k{\cdot}\epsilon^i = 0$, for $i = 1, 2$. The ϵ's are also called *polarisation vectors*. Eq. (5.46) now reads with $\alpha_\pm = \frac{1}{\sqrt{2}}\left(\alpha_0 \pm \alpha_3\right)$

$$\alpha_+ \left|\text{physical states}\right\rangle = 0\,, \tag{5.50}$$

with $\alpha_0 + \alpha_3 \simeq k^\mu a_\mu$. In the frame with $(k^\mu) = \left(k^0, 0, 0, k^0\right)$ we have

$$\left(\epsilon^\lambda\right)_\mu = \delta_\mu^\lambda\,. \tag{5.51}$$

The α's have the same commutation relations as the a's, as we have used unitary rotations (see also Eq. (5.48) and Eq. (5.49)). It follows, with $i = 1, 2$:

$$\begin{aligned} \left[\alpha_i(\mathbf{k}), \alpha_i^\dagger(\mathbf{k}')\right] &= (2\pi)^3\, \delta(\mathbf{k} - \mathbf{k}') \\ \left[\alpha_+(\mathbf{k}), \alpha_-^\dagger(\mathbf{k}')\right] &= -(2\pi)^3\, \delta(\mathbf{k} - \mathbf{k}') \\ \left[\alpha_\pm(\mathbf{k}), \alpha_\pm^{(\dagger)}(\mathbf{k}')\right] &= 0 = \left[\alpha_\pm(\mathbf{k}), \alpha_i^{(\dagger)}(\mathbf{k}')\right]. \end{aligned} \tag{5.52}$$

Let us now examine the **physical Hilbert space** $\mathcal{H}$. It is the physical subspace $\mathcal{F}_{\text{phys}} \subset \mathcal{F}$ with

$$|\Psi\rangle \in \mathcal{F}_{\text{phys}} \Rightarrow \alpha_+ |\Psi\rangle = 0\,. \tag{5.53}$$

It follows

$$|\Psi\rangle \in \mathcal{F}_{\text{phys}} \Rightarrow \alpha_i^\dagger |\Psi\rangle \in \mathcal{F}_{\text{phys}}, \quad \text{for} \quad i = 1, 2\,, \tag{5.54}$$

with

$$\alpha_+ \alpha_i^\dagger |\Psi\rangle = \alpha_i^\dagger \alpha_+ |\Psi\rangle = 0\,. \tag{5.55}$$

[1] unitary rotations keep the canonical commutation relations

Also

$$a_+^\dagger |\Psi\rangle \in \mathcal{F}_{\text{phys}}\,, \tag{5.56}$$

with

$$\alpha_+ \alpha_+^\dagger |\Psi\rangle = \alpha_+^\dagger \alpha_+ |\Psi\rangle = 0. \tag{5.57}$$

This indicates, that everything, that commutes with α_+, is in the physical subspace. Therefore

$$\alpha_-^\dagger |\Psi\rangle \notin \mathcal{F}_{\text{phys}}\,, \tag{5.58}$$

since

$$\begin{aligned}\alpha_+ \alpha_-^\dagger |\Psi\rangle &= \alpha_-^\dagger \alpha_+ |\Psi\rangle + \left[\alpha_+, \alpha_-^\dagger\right] |\Psi\rangle \\ &= \left[\alpha_+, \alpha_-^\dagger\right] |\Psi\rangle \sim |\Psi\rangle \neq 0\,. \end{aligned}\tag{5.59}$$

We conclude, that

$$\boxed{\mathcal{F}_{\text{phys}} = \text{span}\left[\left(a_+^\dagger\right)^{n_+} \left(a_1^\dagger\right)^{n_1} \left(a_2^\dagger\right)^{n_2} |0\rangle\right]\,.} \tag{5.60}$$

$\mathcal{F}_{\text{phys}}$ contains only states with semi-positive norm

$$\langle\Psi|\Psi\rangle \geq 0\,. \tag{5.61}$$

Indeed, it is

$$\begin{aligned}\|\, \alpha_+^\dagger |\Psi\rangle \,\|^2 &= \langle\Psi|\, \alpha_+ \alpha_+^\dagger |\Psi\rangle \\ &= \langle\Psi|\, \alpha_+^\dagger \alpha_+ |\Psi\rangle = 0\,,\end{aligned}\tag{5.62}$$

and

$$\|\left(\alpha_1^\dagger\right)^{n_1} \left(\alpha_2^\dagger\right)^{n_2} |0\rangle\,\| > 0\,, \tag{5.63}$$

with $\left[\alpha_i^\dagger, \alpha_i\right] = +(2\pi)^3\, 2k^0 \delta$. If we identify two states $|\Psi_1\rangle\,, |\Psi_2\rangle$ with $\|\,|\Psi_1\rangle - |\Psi_2\rangle\,\| = 0$, every matrix element of an operator $O\left(\alpha_i^{(\dagger)}, \alpha_+^{(\dagger)}\right)$ vanishes, and $\langle\Psi|\, O\,(|\Psi_1\rangle - |\Psi_2\rangle)$ vanishes. This means, that we define the physical Hilbert space as the space of equivalence classes

$$\mathcal{H} = \mathcal{F}_{\text{phys}}/\!\sim\,, \tag{5.64}$$

with $|\Psi_1\rangle \sim |\Psi_2\rangle$ for $\|\,|\Psi_1\rangle - |\Psi_2\rangle\,\| = 0$. For $|\Psi\rangle \in \mathcal{H}$, we have

$$\langle\Psi|\Psi\rangle > 0, \quad \text{for} \quad |\Psi\rangle \neq 0$$

$$\alpha_+ |\Psi\rangle = 0\,, \tag{5.65}$$

and hence the EOMs are satisfied, since

$$\langle\Psi'\big|\, \partial_\mu F^{\mu\nu} |\Psi\rangle = \langle\Psi'\big|\, \partial_\nu \partial_\mu A^\mu |\Psi\rangle = 0\,. \tag{5.66}$$

We can now introduce the **Feynman rules** for gauge fields. For the propagator we find, for $x^0 > y^0$

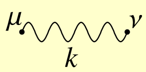
$$
\begin{aligned}
\langle 0|\, A_\mu(x)\, A_\nu(y)\, |0\rangle &= \langle 0| \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}} \int \frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\,(k')^0}}\, \mathrm{e}^{-\mathrm{i}kx+\mathrm{i}k'y} \left[a_\mu(\mathbf{k}), a_\nu^\dagger(\mathbf{k}')\right] |0\rangle \\
&= \langle 0| \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}} \int \frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\,(k')^0}}\, \mathrm{e}^{-\mathrm{i}kx+\mathrm{i}k'y}\, (2\pi)^3\, \delta(\mathbf{k}-\mathbf{k}')\, |0\rangle \\
&= -\eta_{\mu\nu} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2k^0}\, \mathrm{e}^{-\mathrm{i}k(x-y)}\,, \qquad (5.67)
\end{aligned}
$$

where similar to Eq. (4.103) the last integral corresponds to the scalar propagator $\mathcal{D}_F(x-y)$ (Eq. (3.85)). With analogous arguments we find the

Feynman-propagator (for gauge fields)

$$\langle 0|\, T\, A_\mu(x)\, A_\nu(y)\, |0\rangle = -\eta_{\mu\nu}\, \mathcal{D}_F(x-y)\,, \qquad (5.68)$$

or



$$= -\frac{\mathrm{i}\,\eta_{\mu\nu}}{k^2+\mathrm{i}\epsilon}\,.$$

Initial and final states are given by

$$|\mathbf{k},\epsilon\rangle = \sqrt{2k^0}\, \alpha^\dagger(\mathbf{k})\, |0\rangle\ . \qquad (5.69)$$

Note, that $\alpha^\dagger = \epsilon^*_\mu \left(a^\dagger\right)^\mu$ (Eq. (5.49)). Hence, we have

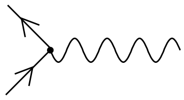
$$
\begin{aligned}
A_\mu\Big|_{\text{annihil.}} |\mathbf{k},\epsilon\rangle &= \int \frac{\mathrm{d}^3k'}{(2\pi)^3}\sqrt{\frac{2k^0}{2\,(k')^0}}\, \mathrm{e}^{\mathrm{i}k'x}\, a_\mu(\mathbf{k}')\, \alpha^\dagger(\mathbf{k}')\, |0\rangle \\
\text{(drop phase)} \quad \rightarrow \quad &\simeq \epsilon^*_\mu(k)\,. \qquad (5.70)
\end{aligned}
$$

That is

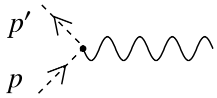
$$
\begin{aligned}
A\, |\mathbf{k},\epsilon\rangle &:= \epsilon^* \\
\langle \mathbf{k},\epsilon|\, A &= \epsilon\,. \qquad (5.71)
\end{aligned}
$$

At the vertices we have:

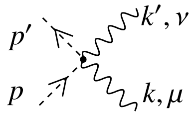
a) $\mathcal{L}_I = e\,\bar{\Psi}\,A\!\!\!/\,\Psi$:

 $= -\mathrm{i}\,e\,\gamma^\mu$

b) $\mathcal{L}_I = D_\mu\phi\;(D^\mu\phi)^* - \partial_\mu\phi\,\partial^\mu\phi^*$:



$= -\mathrm{i}\,e\,\left(p_\mu + p'_\mu\right)$,



$= 2\mathrm{i}\,e^2\,\eta^{\mu\nu}$.

(5.72)

Let us next discuss **gauge independence and Feynman rules**. We can add a longitudinal part to the field A_μ, without changing the physics:

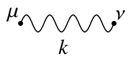
$$A_\mu \quad\rightarrow\quad A_\mu + \alpha\partial_\mu \frac{1}{\partial^\rho\partial^\rho}\partial^\nu A_\nu\,, \tag{5.73}$$

or in the Lagrangian

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \,. \tag{5.74}$$

Then the propagator is

$\langle 0|\,T\;A_\mu A_\nu\,|0\rangle\,(k)$

$=$ 

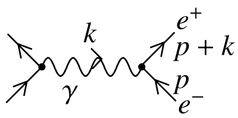
$= -\mathrm{i}\left(\frac{\eta_{\mu\nu}}{k^2+\mathrm{i}\epsilon} - (1-\xi)\frac{k_\mu\,k_\nu}{(k^2+\mathrm{i}\epsilon)^2}\right)$

$$= -\frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu\,k_\nu}{k^2+\mathrm{i}\epsilon}\right). \tag{5.75}$$

The equation of motion in Fourier space with the Lagrangian in Eq. (5.74) is

$$\left(k^2\eta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right)A_\nu(k) = 0\,. \tag{5.76}$$

Note, that for $\xi = 1$ we have the Feynman gauge, $\xi = 0$ the Landau gauge and for $\xi = \infty$ the unitary gauge. When considering the scattering amplitudes, ξ drops out. E.g.:



$\simeq \langle 0|\,T\;A_\mu A_\nu\,|0\rangle\,(k)\,\bar{v}(p+k)\,\gamma^\mu\,u(p)\,.$

This only holds on-shell, i.e. internal Feynman diagrams are ξ dependent. We have used, that:

$$\begin{aligned}
\xi\,k_\mu\,k_\nu\,\bar{v}(p+k)\,\gamma^\mu\,u(p) \quad &= \quad \xi\,k_\nu\,\bar{v}(p+k)\,k\!\!\!/\,u(p) \\
\text{Eq. (4.75)} \quad \rightarrow \quad &= \quad \xi\,k_\nu\,\bar{v}(p+k)\,(k\!\!\!/ + p\!\!\!/ - p\!\!\!/)\,u(p) \\
\text{Eq. (4.75)} \quad \rightarrow \quad &= \quad \xi\,k_\nu\,\bar{v}(p+k)\,(k\!\!\!/ + p\!\!\!/ - m)\,u(p)\,.
\end{aligned} \tag{5.77}$$

We now look at **gauge invariant observables**, for example E and B-fields. We have

$$\begin{aligned} E^i &= -F^{0i} = -\left(\partial^0 A^i - \partial^i A^0\right) \\ B^i &= \epsilon^{ijk} F_{jk}\,. \end{aligned} \tag{5.78}$$

Using Eq. (5.47) they read

$$\begin{aligned} \mathbf{E} &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2k^0}\,\mathrm{i}k^0\Big[\left(\mathbf{a}-\frac{\mathbf{k}}{k^0}\,a_0\right)\mathrm{e}^{-\mathrm{i}kx}-\left(\mathbf{a}^\dagger-\frac{\mathbf{k}}{k^0}\,a_0^\dagger\right)\mathrm{e}^{\mathrm{i}kx}\Big] \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2k^0}\,\mathrm{i}k^0\Big[(\epsilon_1\,\alpha_1+\epsilon_2\,\alpha_2)\,\mathrm{e}^{-\mathrm{i}kx}-\left(\epsilon_1\,\alpha_1^\dagger+\epsilon_2\,\alpha_2^\dagger\right)\mathrm{e}^{\mathrm{i}kx}\Big]-\ldots \\ \sim 0 \quad\to\qquad & \ldots-\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2k^0}\,\mathrm{i}k^0\Big[\frac{\mathbf{k}}{k^0}\,\alpha_+\,\mathrm{e}^{-\mathrm{i}kx}-\frac{\mathbf{k}}{k^0}\,\alpha_+^\dagger\,\mathrm{e}^{\mathrm{i}kx}\Big], \end{aligned} \tag{5.79}$$

with the physical polarisations $\epsilon_{1,2}$. Analogously, we find

$$B^i(x) = \epsilon^{ijl}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\,\mathrm{i}k^j\Big[\epsilon^l\,\alpha\,\mathrm{e}^{-\mathrm{i}kx}-\epsilon^l\,\alpha^\dagger\,\mathrm{e}\mathrm{i}kx\Big]. \tag{5.80}$$

It follows that only $\alpha_{1,2}$ and α_+ appear in **E** and **B**. Sandwiched between physical states $|\Psi\rangle \in \mathcal{H}$, α_+ drops out. The Hamiltonian reads, with $\Pi^i = E^i$

$$\begin{aligned} H &= \int \mathrm{d}^3x\left[\mathbf{\Pi}\,(\partial_0\mathbf{A}-\nabla A_0)+\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \\ &= \int \mathrm{d}^3x\left[\frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)+\nabla\,(\mathbf{E}\,A_0)\right] \\ \nabla\mathbf{E}=0 \quad\to\qquad &= \frac{1}{2}\int \mathrm{d}^3x\,\left(\mathbf{E}^2+\mathbf{B}^2\right), \end{aligned} \tag{5.81}$$

where we have used

$$\begin{aligned} \nabla\mathbf{E} &= \left(-\partial^0\partial^i A^i+\left(\partial^i\right)^2 A^0\right) \\ \partial_\mu A^\mu = 0 \quad\to\qquad &= \Big[-\left(\partial^0\right)^2+\left(\partial^i\right)^2\Big]A^0 = 0\,. \end{aligned} \tag{5.82}$$

We insert the E-B-field operators Eq. (5.79) and Eq. (5.80) and arrive at

$$\begin{aligned} P^0 &= H \simeq \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{k_0}{2k^0}\,k^0\sum_{i=1}^{2}\left(\alpha_i\,\alpha_i^\dagger+\alpha_i^\dagger\,\alpha_i\right) \\ &\simeq \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{k_0}{2k^0}\,k^0\sum_{i=1}^{2}\alpha_i^\dagger(\mathbf{k})\,\alpha_i(\mathbf{k})\,, \end{aligned} \tag{5.83}$$

where we have dropped the α_+-terms in the first line, and the vacuum terms in the second line. Similarly, we get for **P**

$$\begin{aligned} \mathbf{P} &= \int \mathrm{d}^3x\,\mathbf{E}\times\mathbf{B} \\ &\simeq \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathbf{k}\sum_{i=1}^{2}\alpha_i^\dagger(\mathbf{k})\,\alpha_i(\mathbf{k})\,, \end{aligned} \tag{5.84}$$

where we have dropped the vacuum terms.

6. QED

In this chapter we discuss Quantum Electro Dynamics (QED) as an application. We use the following notation:

Dirac fields	electrons, positrons:	e^-, e^+	Ψ_e
Leptons	myons:	μ^-, μ^+	Ψ_μ
	tau:	τ^-, τ^+	Ψ_τ
Gauge field	photons:	γ	A_μ

Note, that the photon is the gauge boson of the U(1)-symmetry, with the Noether charge being the electric charge (see chapter 5).

I. Action and Feynman rules

The action is a sum of the Dirac actions of e, μ, τ and the gauge field action of the photon (see Eq. (5.18))

$$S_{\text{QED}}[A, \Psi_e, \Psi_\mu, \Psi_\tau] = S_D[A, \Psi_e] + S_D[A, \Psi_\mu] + S_D[A, \Psi_\tau] + S_A[A] + S_{gf}[A]\,, \tag{6.1}$$

with the Dirac actions

$$S_D[A, \Psi_{e,\mu,\tau}] = \int \mathrm{d}^4x\, \bar{\Psi}_{e,\mu,\tau} \left(\mathrm{i}\,\not{D} - m_{e,\mu,\tau}\right) \Psi_{e,\mu,\tau}, \quad \text{with} \quad D_\mu = \partial_\mu - \mathrm{i}eA_\mu\,, \tag{6.2}$$

and the gauge field action

$$S_A[A] = -\frac{1}{4} \int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu}, \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{6.3}$$

The gauge fixing term $S_{gf}[A]$ in the covariant gauge is

$$S_{gf}[A] = -\frac{1}{2\xi} \int \mathrm{d}^4x \left(\partial_\mu A^\mu\right)^2\,, \tag{6.4}$$

with the gauge fixing parameter ξ. The gauge transformations are

$$\begin{aligned} \Psi(x) \quad &\to \quad \mathrm{e}^{\mathrm{i}\alpha(x)}\,\Psi(x) = \Psi^\alpha(x) \\ A_\mu(x) \quad &\to \quad A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) = A^\alpha_\mu(x)\,, \end{aligned} \tag{6.5}$$

with

$$S_{\text{QED}}[A^\alpha, \Psi^\alpha] = S_{\text{QED}}[A, \Psi] + \frac{1}{\xi}\frac{1}{e} \int \mathrm{d}^4x\, \partial_\mu A^\mu \partial_\rho \partial^\rho \alpha\,, \tag{6.6}$$

where

$$\Psi = \begin{pmatrix} \Psi_e \\ \Psi_\mu \\ \Psi_\tau \end{pmatrix}. \tag{6.7}$$

Next, we consider the Feynman rules. It is

$$S_{\mathrm{QED}} = S_{\mathrm{free}} + S_{\mathrm{I}}\,, \tag{6.8}$$

with

$$S_{\mathrm{free}} = S_A[A] + S_{gf}[A] + \int \mathrm{d}^4x\,\bar{\Psi}(\mathrm{i}\partial\!\!\!/ - m)\Psi\,, \tag{6.9}$$

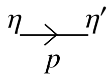
and

$$S_{\mathrm{I}} = e\int \mathrm{d}^4x\,\bar{\Psi}A\!\!\!/\Psi\,. \tag{6.10}$$

We remark, that any other coupling of leptons and photon introduces dimensiona-full couplings to the theory, e.g. spin-coupling

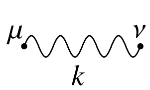
$$\frac{e}{\Lambda}\bar{\Psi}\,\sigma^{\mu\nu}\,\Psi\,F_{\mu\nu}\,\Psi\,, \tag{6.11}$$

where Λ carries momentum dimension one. Such a term makes the theory non-renormalisable. We obtain the propagators for
Leptons (Eq. (4.112)):



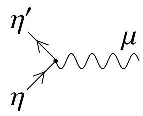
$$= \mathrm{i}\left(\frac{p\!\!\!/ + m_\Psi}{p^2 - m_\Psi^2 + \mathrm{i}\epsilon}\right)_{\eta\eta'}$$

and photon (Eq. (5.75)):



$$= -\frac{\mathrm{i}}{k^2 + \mathrm{i}\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu\,k_\nu}{k^2 + \mathrm{i}\epsilon}\right).$$

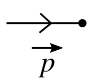
At the vertices it holds (see Eq. (5.72)



$$= -\mathrm{i}\,e\,(\gamma_\mu)_{\eta\eta'}\,. \tag{6.12}$$

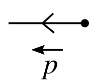
Note, that here the sign is irrelevant, as $A_\mu \to -A_\mu$. Eq. (6.12) has been deduced simply by analogy to the derivation of the scalar self-interaction. Further, we have

incoming lepton:



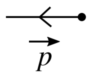
$= u(p)$

outgoing lepton:



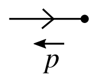
$= \bar{u}(p)$

incoming anti-lepton:



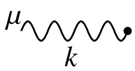
$= \bar{v}(p)$

outgoing anti-lepton:



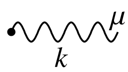
$= v(p)$

incoming photon:



$= \epsilon_\mu(k)$

outgoing photon:



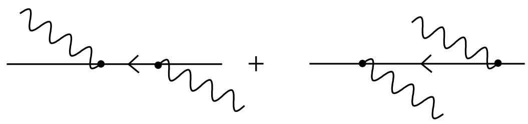
$= \epsilon^*_\mu(k)\,.$ (6.13)

See also Eq. (4.113) ff. and recall the minus sign for fermion loops.

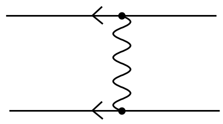
II. Elementary Processes

This section deals with elementary processes. Consider

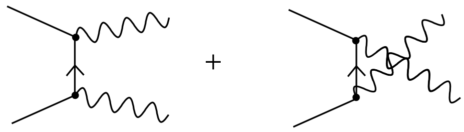
i) Compton scattering: $e^- \gamma \to e^- \gamma$



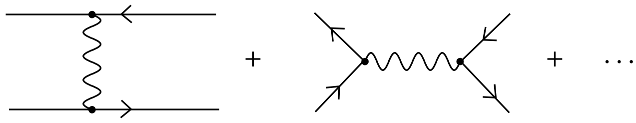
ii) Elastic $e^- e^-$-scattering:



iii) Pain-annihilation/creation: $e^+ e^- \to \gamma\gamma$

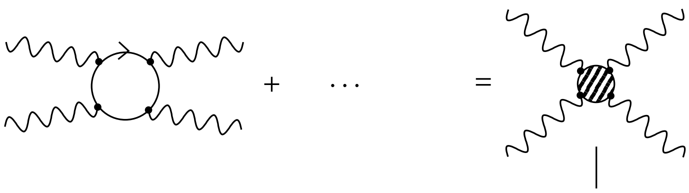


iv) Bhaba-scattering: $e^+ e^- \to e^+ e^-$



tree level processes

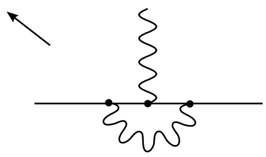
v) light-by-light scattering: (non-linear electrodynamics)



effective four photon vertex

vi) Landé factor (gyromagnetic ratio):

$$\mathrm{i}\not{D} - m_e \to \mathrm{i}\not{D} - m_e + \frac{\Delta g}{2}\frac{e}{4m_e}\sigma_{\mu\nu}F^{\mu\nu}\,, \qquad \Delta g = \frac{\alpha}{\pi}\,,\ \alpha = \frac{e^2}{4\pi}$$

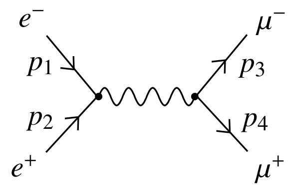


loop effects

Let us compute an example of a tree level process in detail.

Example 15: electron-positron annihilation into muon-antimuon pair ($e^+ e^- \to \mu^+\mu^-$)**.**

We choose this example, because there exists only a single Feynman diagram for this process, namely:



As we look at the highly relativistic case of 2-2 scattering we can use Eq. (3.153):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2s}\,\left|M_{fi}\right|^2 \int \mathrm{d}\Pi_2\,, \tag{6.14}$$

with

$$\int \mathrm{d}\Pi_2 = \frac{1}{2}\frac{s/4}{2(\pi)^2\, 4p_3^0\, p_4^0}\,\mathrm{d}\Omega\,, \quad s = (p_1 + p_2)^2\,. \tag{6.15}$$

We find

$$|M|^2 = \frac{1}{2}\sum_r \frac{1}{2}\sum_{r'}\sum_{s,s'} |M(r, r', s, s')|^2\,, \tag{6.16}$$

where we computed the average by summing over r, r' and summed over all possible splits $s, s,'$. The scattering amplitude is read off by the Feynman rules:

$$\mathrm{i}M = \bar{u}_{\mu_s}(p_3)\left(\mathrm{i}\, e\,\gamma_\rho\right) v_{\mu_{s'}}(p_4)\left[\frac{\eta^{\rho\sigma}}{s}\right]\bar{v}_{e_{r'}}(p_2)\,(\mathrm{i}\, e\,\gamma_\sigma)\, u_{e_r}(p_1)\,, \tag{6.17}$$

where the term in square brackets corresponds to the (on-shell) propagator. Therefore, the gauge fixing parameter ξ drops out (see Eq. (5.77)). The terms in front of and behind the propagator correspond to right- and left-hand side in the diagram, respectively. It follows that

$$|M|^2 = \frac{e^4}{4s}(T_\mu)_{\alpha\beta}\,(T_e)^{\alpha\beta}\,, \tag{6.18}$$

with

$$\begin{aligned}
(T_\mu)_{\alpha\beta} &= \sum_{s,s'} \bar{u}_{\mu_s}(p_3)\,(\mathrm{i}\, e\,\gamma_\alpha)\, v_{\mu_{s'}}(p_4)\cdot\left[\bar{u}_{\mu_s}(p_3)\left(\mathrm{i}\, e\,\gamma_\beta\right) v_{\mu_{s'}}(p_4)\right]^* \\
(T_e)^{\alpha\beta} &= \sum_{r,r'} \bar{v}_{e_{r'}}(p_2)\,(\mathrm{i}\, e\,\gamma^\alpha)\, u_{e_r}(p_1)\cdot\left[\bar{v}_{e_{r'}}(p_2)\left(\mathrm{i}\, e\,\gamma^\beta\right) u_{e_r}(p_1)\right]^*\,.
\end{aligned} \tag{6.19}$$

We use Eq. (4.83), i.e.

$$\begin{aligned}
\sum_s u_{\mu_s}(p_3)\,\bar{u}_{\mu_s}(p_3) &= (\not{p}_3 + m_\mu) \\
\sum_s v_{\mu_{s'}}(p_4)\,\bar{v}_{\mu_{s'}}(p_4) &= (\not{p}_4 - m_\mu)\,,
\end{aligned} \tag{6.20}$$

to compute

$$\sum_{s.s'} \bar{u}_{\mu_s}(p_3)\,\gamma_\alpha\left[v_{\mu_{s'}}(p_4)\,\bar{v}_{\mu_{s'}}(p_4)\right]\gamma_\beta^*\, u_{\mu_s}(p_3) = \mathrm{tr}\left(\not{p}_3 + m_\mu\right)\,\gamma_\alpha\left(\not{p}_4 - m_\mu\right)\,\gamma_\beta\,, \tag{6.21}$$

with

$$\begin{aligned}
\left[\bar{u}_s(p)\gamma_\alpha v_{s'}(q)\right]^* &= v_{s'}^\dagger(q)\,\gamma^0\gamma^0\,\gamma_\alpha^\dagger\,\gamma^0\gamma^0\,\bar{u}_s^\dagger(p)\,, \quad \gamma^0\gamma^0 = \mathbb{1} \\
&= \bar{v}_{s'}^\dagger(q)\,\gamma_\alpha\, u_s(p)\,.
\end{aligned} \tag{6.22}$$

As we consider the highly relativistic limit, we drop m_μ, m_e. Then

$$\begin{aligned} (T_\mu)_{\alpha\beta} &= \operatorname{tr}\left(\not{p}_3 + m_\mu\right)\gamma_\alpha\left(\not{p}_4 - m_\mu\right)\gamma_\beta \\ \operatorname{tr}\gamma^{2n+1} = 0 \quad \rightarrow \quad &= \operatorname{tr}\not{p}_3\gamma_\alpha\not{p}_4\gamma_\beta + \operatorname{tr}\gamma_\alpha\gamma_\beta\, m_\mu^2 . \end{aligned} \tag{6.23}$$

We use

$$\begin{aligned} \operatorname{tr}\gamma^\rho\gamma^\sigma &= \frac{1}{2}\operatorname{tr}\{\gamma^\rho,\gamma^\sigma\} = \frac{1}{2}\operatorname{tr}(2\eta^{\rho\sigma}) = 2\eta^{\rho\sigma} \\ \operatorname{tr}\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta &= 2\eta^{\rho\sigma}\operatorname{tr}\gamma^\alpha\gamma^\beta - \operatorname{tr}\gamma^\sigma\gamma^\rho\gamma^\alpha\gamma^\beta = 8\eta^{\rho\sigma}\eta^{\alpha\beta} - \operatorname{tr}\gamma^\sigma\gamma^\rho\gamma^\alpha\gamma^\beta \\ &= \cdots = 4\left(\eta^{\rho\sigma}\eta^{\alpha\beta} - \eta^{\rho\alpha}\eta^{\beta\sigma} + \eta^{\rho\beta}\eta^{\alpha\sigma}\right), \end{aligned} \tag{6.24}$$

and obtain

$$\begin{aligned} (T_\mu)_{\alpha\beta} &= 4\left(p_{3_\alpha}p_{4_\beta} + p_{3_\beta}p_{4_\alpha} - \eta_{\alpha\beta}p_3p_4\right) - 4\eta_{\alpha\beta}m_\mu^2 \\ s \gg m_\mu^2 \quad \rightarrow \quad &\simeq 4\left(p_{3_\alpha}p_{4_\beta} + p_{3_\beta}p_{4_\alpha} - \eta_{\alpha\beta}p_3p_4\right). \end{aligned} \tag{6.25}$$

Similarly, we get

$$(T_e)^{\alpha\beta} \simeq 4\left(p_1^\alpha p_2^\beta + p_1^\beta p_2^\alpha\right), \tag{6.26}$$

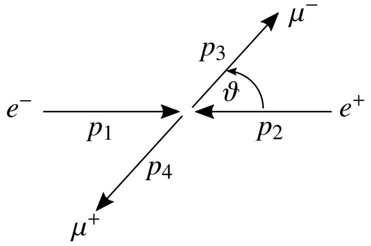
and arrive at

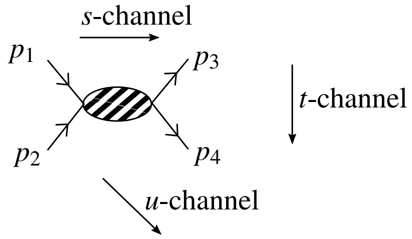
$$|M|^2 = \frac{e^4}{4s^2}\cdot 2\cdot 16\Big[(p_1\cdot p_4)(p_2\cdot p_3)+(p_1\cdot p_3)(p_2\cdot p_4)\Big] = \frac{8e^4}{s^2}\cdot 2\cdot 16\Big[(p_1\cdot p_4)(p_2\cdot p_3)+(p_1\cdot p_3)(p_2\cdot p_4)\Big]. \tag{6.27}$$

In summary, and after inserting Eq. (6.27) in Eq. (6.14), we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\alpha^2}{p_3^0p_4^0 s}\Big[(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4)\Big], \quad \alpha = \frac{e^2}{4\pi}. \tag{6.28}$$

Note, that this expression depends on the scattering angle ϑ. Furthermore, $s = (p_1+p_2)^2$, $t = (p_1-p_3)^2$ and $u = (p_1-p_4)^2$ are called *Mandelstam variables*.





The scattering angle is given by

$$\cos\vartheta = \frac{\mathbf{p}_1\mathbf{p}_3}{|\mathbf{p}_1||\mathbf{p}_3|}. \tag{6.29}$$

In the highly relativistic limit, it holds

$$\begin{aligned} p_1\cdot p_3 &= p_1^0p_3^0 - \mathbf{p}_1\mathbf{p}_3 \simeq \frac{1}{4}s - \frac{1}{4}s\cos\vartheta = \frac{1}{4}s(1-\cos\vartheta) = p_2\cdot p_4 \\ p_1\cdot p_4 &= \frac{1}{4}s(1+\cos\vartheta) = p_2\cdot p_3 \\ \Rightarrow (p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) &= \frac{1}{16}s^2\left(2+2\cos^2\vartheta\right) = \frac{1}{8}s^2\left(1+\cos^2\vartheta\right). \end{aligned} \tag{6.30}$$

The final result for $|M|^2$ is

$$|M|^2 = e^4 \left(1 + cos^2\vartheta\right) = 16\pi^2\alpha^2 \left(1 + cos^2\vartheta\right), \quad \alpha = \frac{e^2}{2\pi}. \tag{6.31}$$

If we compare Eq. (6.31) with that for scalar 2-2 scattering (Eq. (3.40)), $|M|^2 = \lambda^2$, we see, that for fermions the scattering angle is important, whereas for scalar fields it is not. Inserting Eq. (6.31) in Eq. (6.14) and using $4p_3^0 p_4^0 \simeq s$ yields the cross section

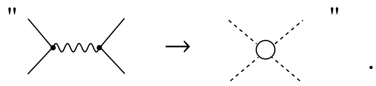
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(1 + \cos\vartheta). \tag{6.32}$$

Again, compare this to the cross section of scalar 2-2 scattering (Eq. (3.153)), i.e. $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(4\pi)^2}\frac{\lambda^2}{4\pi}$.

We remark, that the intermediate virtual photon was chosen in the Feynman gauge, i.e. $\xi = 1$. However, we have shown in Eq. (5.77), that any choice of ξ leads to the same result, in particular $\xi = 0$. Further, in the high energy limit also $(p_1 - p_2)_\mu \bar{v}(p_2)\,\gamma^\mu\, u(p_1) \overset{\sim m_e}{\approx} 0$. Only the physical polarisations ϵ_1 and ϵ_2 play a role, ϵ_3 drops out (see Eq. (5.48)). This argument also applies to $\bar{u}(p_3)\,\gamma^\nu\, v(p_4)$. In summary we have

$$\begin{aligned} \bar{u}(p_3)\,\gamma^\nu\, v(p_4)\,(p_{3/4})_\nu &\approx 0 \\ \bar{v}(p_2)\,\gamma^\mu\, u(p_1)\,(p_{1/2})_\mu &\approx 0\,. \end{aligned} \tag{6.33}$$

So if $p_{3,4}$ are orthogonal to the beam axis, defined by $p_{1/2}$, the related polarisation ϵ_1 or ϵ_2 also 'drops out of the game'. In this case, $\alpha = \pi/2$, only one polarisation contributes to the scattering, for $\alpha = 0$, both. Lastly, note, that in the highly relativistic case and for $\alpha = \pi/2$:



7. Renormalisation

As we have seen in the previous chapters, loop diagrams are divergent. In this chapter we discuss renormalisation, a mathematical approach, to cancel singularities from the integrals.

I. ϕ^4-theory

In ϕ^4-theory the action is given by (see Eq. (3.21))

$$S[\phi] = -\frac{1}{2}\int d^4x\,\phi_0\left(\partial^2 + m_0^2\right)\phi_0 - \frac{\lambda_0}{4!}\int d^4x\,\phi_0^4\,, \tag{7.1}$$

with *bare* fields ϕ_0 and parameters/couplings m_0^2 and λ_0. We write

$$\begin{aligned}\phi_0 &= Z_\phi^{1/2}\,\phi \\ m_0^2 &= Z_m\,m^2 \\ \lambda_0 &= Z_\lambda\,\lambda\,,\end{aligned} \tag{7.2}$$

with *renormalised* or physical fields ϕ, parameters m^2, λ and *multiplicative* renormalisations Z_ϕ, Z_m, Z_λ. The Z's are expanded in powers of λ:

$$Z = 1 + \delta Z\,, \quad \delta Z = \delta Z_1\lambda + \delta Z_2\lambda^2 + \dots\,, \tag{7.3}$$

where the first part corresponds to classical theory and δZ to quantum corrections. Recalling LSZ-formalism we use Eq. (3.166) with fields ϕ_0

$$\langle T\,\phi_0\,\phi_0\rangle(p)\big|_{\text{pole}} = \frac{iZ}{p^2 - m_{\text{phys}}^2} + \text{finite} = Z_\phi\,\langle T\,\phi\,\phi\rangle\big|_{\text{pole}}\,. \tag{7.4}$$

We demand $Z_\phi = Z$, which implies

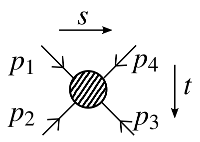
$$\langle T\,\phi\,\phi\rangle\big|_{\text{pole}} = \frac{i}{p^2 - m^2} + \text{finite}\,. \tag{7.5}$$

Here, we have implicitly fixed Z_ϕ such that $m^2 = m_{\text{phys}}^2$, i.e. $p^2 = m^2$. Eq. (7.4) and Eq. (7.5) can be cast into

renormalisation conditions (1-2)

$$\begin{aligned}\Big[\langle T\,\phi\phi\rangle(p)\Big]^{-1}_{p^2=m^2} &\overset{!}{=} 0 \\ \partial_{p^2}\Big[\langle T\,\phi\phi\rangle(p)\Big]^{-1}_{p^2=m^2} &\overset{!}{=} 1\,.\end{aligned} \tag{7.6}$$

This fixes the constants Z_ϕ and Z_m. More generally, we fix $\langle T\,\phi\phi\rangle$ at some scale $p^2 = \mu^2$, where μ is called renormalisation scale. The coupling renormalisation Z_λ is fixed, by fixing the amputated four-point function:



$$\Bigg|_{s=t=u=m^2} \stackrel{!}{=} -\mathrm{i}\,\lambda\,.$$

If we write this in terms of the Green function (using Eq. (7.5)), we obtain the third

> **renormalisation condition** (3)
>
> $$\prod_i \Big[\langle T\,\phi\,\phi\rangle\,(p_i) \Big]^{-1} \cdot \langle T\,\phi(p_1)\cdots\phi(p_4)\rangle \Big|_{s=t=u=m^2} = -\mathrm{i}\,\lambda\,, \tag{7.7}$$

where $\lambda = \lambda_{\text{phys}}\big|_{\text{symmetric point}}$. The renormalisation conditions Eq. (7.6) and Eq. (7.7) fix the map between the bare quantities ϕ_0, m_0, λ_0 to the renormalised (finite) quantities ϕ, m, λ. The finiteness of correlation functions of the renormalised fields ϕ follows from the finiteness of Eq. (7.6) and Eq. (7.7). Hence, the Z's have to cancel the loop divergences. Thus, the Z's carry the singularities. Note, that in (perturbatively) renormalisable theories, it is sufficient to introduce the Z's (and similar quantities) for getting a manifestly finite theory. The freedom of (re)-normalising fields and couplings also encodes, that Green functions are *not* by themselves physical observables. For example, we could have renormalised the theory at some other momentum scale $p^2 = \mu^2$ with the renormalisation conditions, with

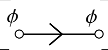
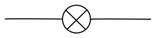
$$\begin{aligned} \lambda &= \lambda_{\text{phys}}\big|_{p^2=\mu^2} \\ m^2 &= m^2_{\text{phys}}\big|_{p^2=\mu^2}\,. \end{aligned} \tag{7.8}$$

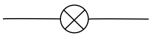
Physics is invariant under changing μ, which is expressed in the

> **renormalisation group equation**
>
> $$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}(\text{phys. observables}) = 0\,. \tag{7.9}$$

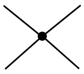
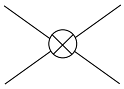
Note, that the renormalisation conditions encode the **reparametrisation invariance of the theory** and the **insensitivity of physics** to the specific renormalisation scheme. μ is called *renormalisation group (RG) scale*. We remark, that the generator of the RG is $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}$, and the RG is a one-parameter, Abelian semi group (see QFT II).
Let us now formulate the **Feynman rules** in terms of renormalised quantities (where we have dropped the $i\epsilon$):

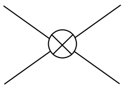
Propagator:

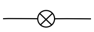
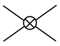
$\Big[$  $\Big]^{-1}$ $= Z_\phi \, \frac{p^2 - Z_m\, m^2}{\mathrm{i}} = \left[\frac{\mathrm{i}}{p^2 - m^2}\right]^{-1} -$  ,

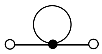
where  $= (-\mathrm{i}) \Big[\underbrace{(1 - Z_\phi)}_{<1} \, p^2 - (1 - Z_\phi\, Z_m)\, m^2\Big].$

Vertex:

 $= -\mathrm{i}\, Z_\lambda\, Z_\phi^2\, \lambda = -\mathrm{i}\, \lambda +$  ,

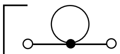
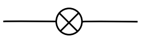
where  $= \mathrm{i}\, \lambda\, (1 - Z_\phi^2\, Z_\lambda)\,.$

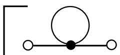
Note, that  ,  are called counter terms. Z_ϕ, Z_m, Z_λ cancel singularities, that are proportional to p^2, m^2 and λ, respectively. Next, we examine **renormalisation at one loops**. First, we consider the mass correction (see Eq. (3.184)).

 = +  + $\mathcal{O}(\lambda^2)$

$$= \frac{\mathrm{i}}{p^2 - m^2} + \frac{\mathrm{i}}{p^2 - m^2}\Big[-\mathrm{i}\,\Pi(p)\Big]\frac{\mathrm{i}}{p^2 - m^2} + \ldots,$$

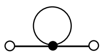
with self-energy:

$-\mathrm{i}\,\Pi(p) = \Bigg[$  +  $\Bigg]$

$= \Bigg[$  $+ \mathrm{i}\,(1 - Z_\phi)\, p^2 + \mathrm{i}\,(1 - Z_\phi\, Z_m)\, m^2 \Bigg]$.

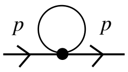
finite

For the loop diagram, we have:

 $= -\mathrm{i}\, \lambda \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \frac{\mathrm{i}}{q^2 - m^2}\,.$

Consequently, the self-energy is

$$-\mathrm{i}\,\Pi(p) = -\frac{\mathrm{i}\,\lambda}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \frac{\mathrm{i}}{q^2 - m^2} - \mathrm{i}\,(1 - Z_\phi)\, p^2 + \mathrm{i}\,(1 - Z_\phi\, Z_m)\, m^2\,. \tag{7.10}$$

Note, that  has no dependence on the external momentum p. Therefore, $Z_\phi\big|_{\text{1-loop}} = 1$. Further, it is

$\Bigg[$  $\Bigg]^{-1}\Bigg|_{p^2=m^2} = 0\,.$

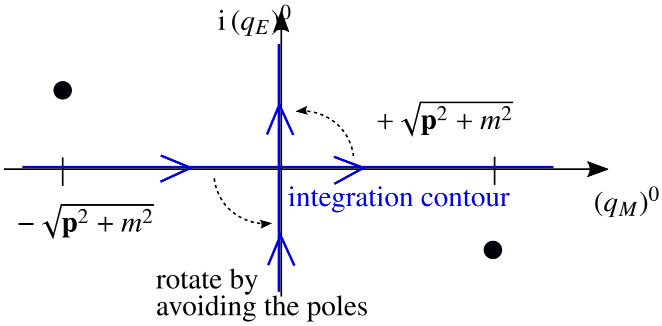


Figure 7.1.: Recall the iϵ of time-ordering: $\frac{1}{q^2-m^2+\mathrm{i}\,\epsilon}$. The rotated Euclidean contour runs $\mathrm{i}(q_E)^0$ from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$, or $(q_E)^0$ from $-\infty$ to $+\infty$. As this rotation does not swipe the poles, the integration stays the same.

The renormalisation condition in Eq. (7.6) implies

$$\Rightarrow \Pi(p)\big|_{\text{1-loop}} \hat{=} 0\,. \tag{7.11}$$

We conclude, that

$$\boxed{1 - Z_m = \frac{1}{2}\frac{\lambda}{m^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{i}}{q^2-m^2}\,. \tag{7.12}}$$

It remains to compute

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{i}}{q^2-m^2+\mathrm{i}\epsilon}\,, \tag{7.13}$$

where we have to encounter two problems, namely that the integrand diverges on-shell ($q^2 = m^2$), and that the integral diverges for $q^2 \to \infty$. The diverging integrand can be resolved with **Wick rotation**. For this, we rescale momentum in Minkowski space to Euclidean space as shown in figure 7.1. Then

$$\begin{aligned}
& & (q_M)^0 &= \mathrm{i}\,(q_E)^0 \\
(\eta_E)^{\mu\nu} = -\mathbb{1} &\quad\to & &\Rightarrow\ (q_M)_\mu (q_M)^\mu = -(q_E)_\mu (q_E)^\mu = -(q_E)_\mu (q_E)_\mu \\
& & &\Rightarrow\ \int_{\mathbb{R}^4} \mathrm{d}^4 q_M = \mathrm{i}\int_{\mathbb{R}^4} \mathrm{d}^4 q_E \\
& & &\Rightarrow\ \int \frac{\mathrm{d}^4 q_M}{(2\pi)^4}\frac{\mathrm{i}}{q_M^2 - m^2} = \int \frac{\mathrm{d}^4 q_E}{(2\pi)^4}\frac{\mathrm{i}}{q_E^2+m^2}\,.
\end{aligned} \tag{7.14}$$

The divergent integral for $q^2 \to \infty$ is dealt with by **regularisation**. Consider for example a momentum cut-off Λ:

$$\int_{\mathbb{R}^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \to \int_{q^2\leq\Lambda^2} \frac{\mathrm{d}^4 q}{(2\pi)^4}\,. \tag{7.15}$$

Then

$$\int\limits_{q^2\le\Lambda^2} \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2} = \int\limits_0^\Lambda \frac{dq}{(2\pi)^4}\,q^3\int d\Omega\,\frac{1}{q^2+m^2} = \frac{1}{8\pi^2}\int\limits_0^\Lambda dq\,\frac{q^3}{q^2+m^2}$$

$$= \frac{1}{16\pi^2}\left[\Lambda^2+m^2\ln\frac{m^2}{\Lambda^2+m^2}\right]. \tag{7.16}$$

Another example is dimensional regularisation. We rewrite the four-dimensional integral as

$$\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2} = \left[\left(\bar\mu^2\right)^{\frac{4-d}{2}}\int\frac{d^dq}{(2\pi)^d}\right]\frac{1}{q^2+m^2}$$

$$= \frac{\Omega_d}{(2\pi)^d}\left(\bar\mu^2\right)^{\frac{4-d}{2}}\int\limits_0^\infty dq\,q^{d-1}\frac{1}{q^2+m^2}, \tag{7.17}$$

with the angular volume Ω_d. Note, that the term in square brackets in the first line has dimension 4 due to the scaling factor in front of the d-dimensional integral. For $d<2$ the integral in the last line is finite, and we can compute Eq. (7.17), and then analytically extend the result. We use

$$\int\frac{d\Omega_d}{(2\pi)^d}: \quad \sqrt{\pi}^d = \left(\int dx\,e^{-x^2}\right)^d = \int d^dx\,e^{-\mathbf{x}^2}, \quad \mathbf{x}=\begin{pmatrix}x_1\\ \vdots\\ x_d\end{pmatrix}$$

$$= \int d\Omega_d\int\limits_0^\infty dx\,x^{d-1}\,e^{-x^2} = \int d\Omega_d\,\Gamma\left(\frac{d}{2}\right)$$

$$\Rightarrow\quad \Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}, \tag{7.18}$$

and

$$\int\limits_0^\infty dq\,q^{d-1}\frac{1}{(q^2+m^2)^n} = \frac{1}{2}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{m^2}\right)^{n-\frac{d}{2}} \tag{7.19}$$

$$\boxed{\Rightarrow\int\limits_0^\infty\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+m^2)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{m^2}\right)^{n-\frac{d}{2}}.} \tag{7.20}$$

With this, we obtain

$$\left(\bar\mu^2\right)^{\frac{4-d}{2}}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+m^2} = \frac{1}{(4\pi)^{\frac{d}{2}}}\Gamma\left(1-\frac{d}{2}\right)m^2\left(\frac{\bar\mu^2}{m^2}\right)^{2}-\frac{d}{2}. \tag{7.21}$$

Then, for $d=4-2\epsilon$ with $\epsilon\to 0$, this is

$$\left(\bar\mu^2\right)^{\epsilon}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+m^2} = \frac{m^2}{16\pi^2}\left[\frac{1}{\epsilon}+\gamma-1+\ln 4\pi-\ln\frac{m^2}{\bar\mu^2}\right], \tag{7.22}$$

with

$$\begin{aligned}\Gamma(-1+\epsilon) &= \frac{1}{-1+\epsilon}\Gamma(\epsilon) \quad \leftarrow \quad x\Gamma(x) = \Gamma(x+1) \\ &= -\frac{1}{\epsilon} + \gamma - 1 + O(\epsilon)\,,\end{aligned} \tag{7.23}$$

and the Euler-Mascheroni constant $\gamma = 0.577\ldots$. This allows us to determine $Z_m\big|_{\text{1-loop}}$. With cut-off regularisation (Eq. (7.16)) we get

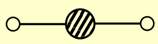
$$Z_m = 1 - \frac{1}{2}\frac{1}{16\pi^2}\lambda\left(\frac{\Lambda^2}{m^2} + \ln\frac{1}{1+\frac{\Lambda^2}{m^2}}\right). \tag{7.24}$$

And with dimensional regularisation (Eq. (7.23)) we obtain

$$Z_m = 1 - \frac{1}{2}\frac{1}{16\pi^2}\lambda\left(-\frac{1}{\epsilon} + \gamma - 1 + \ln 4\pi - \ln\frac{m^2}{\bar{\mu}^2}\right). \tag{7.25}$$

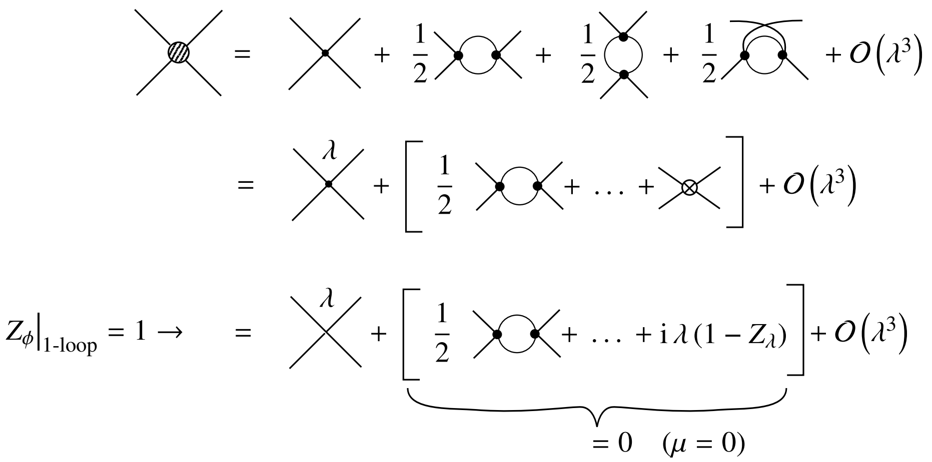
Note, that in Eq. (7.24) and Eq. (7.25) the term $\frac{1}{16\pi^2}\lambda$ is the expansion coefficient in ϕ^4-theory. The equivalence between these equations is best seen with: $\ln\frac{1}{1+\frac{\Lambda^2}{m^2}} = \ln\frac{m^2}{\Lambda^2} + \ln\frac{1}{1+\frac{m^2}{\Lambda}}$.

Finally, in both cases at one loop we have



$$= \frac{\mathrm{i}}{p^2 - m^2} + O(\lambda^2)\,. \tag{7.26}$$

Next, we have to consider the coupling correction



The renormalisation condition for $t = s = u = 0$, i.e. $\mu^2 = 0$ becomes

$$1 - Z_\lambda = \frac{3\lambda}{2}\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{i}}{(q^2-m^2)^2}\,. \tag{7.27}$$

Using Wick rotation (Eq. (7.20)) with dimensional regularisation ($n = 2$, $2 - \frac{d}{2} = \epsilon$), we compute

$$
\begin{aligned}
-\frac{3\lambda}{2}\mu^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2+m^2)^2} &= -\frac{3\lambda}{2} \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\epsilon)}{\Gamma(2)} \left(\frac{m^2}{\mu^2}\right)^{-\epsilon} \\
&= -\frac{3\lambda}{2} \frac{1}{16\pi^2} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln \frac{m^2}{\mu^2}\right), \qquad (7.28)
\end{aligned}
$$

where in the last line, we used the expansion: $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon)$. In summary, we state, that (at renormalisation group scale $\mu^2 = 0$)

$$
Z_\lambda = 1 + \frac{3}{2} \frac{\lambda}{16\pi^2} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln \frac{m^2}{\mu^2}\right). \qquad (7.29)
$$

With $Z_\phi = 1$ our theory is consistent at one loop. Also, it is renormalisable at one loop. We remark, that the renormalised correlation functions $\langle \phi(p_1)\, \phi(p_2) \rangle_{\text{1-loop}}$, $\langle \phi(p_1) \cdots \phi(p_4) \rangle_{\text{1-loop}}$ are *finite*, but depend on the renormalisation scale μ. Higher correlation functions at one loop are finite from the onset, e.g. the six-point function, as at $p_i = 0$:

$$
\langle \phi_0(p_1) \cdots \phi_0(p_6) \rangle_{\text{1-loop}} \sim \lambda^3 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2)^3}, \qquad (7.30)
$$

is finite. Note, that a singularity in $\langle \phi_0(p_1) \cdots \phi_0(p_6) \rangle_{\text{1-loop}}$ would be disastrous, because there is no counter term for it. Hence, perturbative renormalisability (in ϕ^4-theory) implies, that all correlation functions *to all orders* in perturbation theory are finite, by adjusting Z_ϕ, Z_m, Z_λ. Also note, that 'Physics' does *not* depend on the renormalisation scheme, which yields the

renormalisation group invariance

$$
\mu \frac{d}{d\mu} \text{observable} = 0. \qquad (7.31)
$$

Moreover, it does *not* depend on the cut-off scale

$$
\Lambda \frac{d}{d\Lambda} \text{observable} = 0. \qquad (7.32)
$$

Evidently, the bare quantities know nothing of the renormalisation point. Hence

$$
\mu \frac{d}{d\mu} \phi_0 = \mu \frac{d}{d\mu} m_0 = \mu \frac{d}{d\mu} \lambda_0 = 0. \qquad (7.33)
$$

It follows, that

$$
\begin{aligned}
\mu \frac{d\phi}{d\mu} \frac{1}{\phi} &= -\frac{1}{2} \mu \frac{dZ_\phi}{d\mu} \frac{1}{Z_\phi} = -\gamma_\phi \\
\mu \frac{d\lambda}{d\mu} \frac{1}{\lambda} &= -\mu \frac{dZ_\lambda}{d\mu} \frac{1}{Z_\lambda} = \beta_\lambda \\
\mu \frac{dm^2}{d\mu} \frac{1}{m^2} &= -\mu \frac{dZ_m}{d\mu} \frac{1}{Z_m} = \gamma_m, \qquad (7.34)
\end{aligned}
$$

which are also referred to as *beta-functions*. In turn, the renormalised quantities are insensitive to the cut-off. Hence

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \phi = \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} m = \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \lambda = 0 \,. \tag{7.35}$$

Therefore, Λ and μ scaling are (asymptotically) directly related.

A. Complementary Calculations

I. Normalisation of the coherent state

Here, the explicit calculations for the normalisation $\mathcal{N}(\alpha)$ of the coherent state $|\alpha\rangle$ are provided. We start with the ansatz given in section III

$$|\alpha\rangle = \frac{1}{\mathcal{N}(\alpha)} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{\infty} \left(\int \frac{d^3 p_i}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p_i}}}} \alpha(\mathbf{p_i}) \right) |\mathbf{p_1} \cdots \mathbf{p_n}\rangle \,, \tag{A.1}$$

with

$$a(\mathbf{p})\, |\mathbf{p_1} \cdots \mathbf{p_n}\rangle = \sum_{i=1}^{n} (2\pi)^3 \sqrt{2\omega_{\mathbf{p_i}}}\, |\mathbf{p_1} \cdots \mathbf{p_{i-1}}\, \mathbf{p_{i+1}} \cdots \mathbf{p_n}\rangle\, \delta(\mathbf{p} - \mathbf{p_i})\,. \tag{A.2}$$

From Eq. (A.2) and with normal ordering (Eq. (2.97)) it follows

$$\begin{aligned}
& \frac{1}{n!}\, a(\mathbf{p}) \left(\int \frac{d^3 p'}{(2\pi)^3}\, \alpha(\mathbf{p}')\, a^\dagger(\mathbf{p}') \right)^n |0\rangle \\
= \;& \frac{1}{n!}\, n\, \alpha(\mathbf{p}) \left(\int \frac{d^3 p'}{(2\pi)^3}\, \alpha(\mathbf{p}')\, a^\dagger(\mathbf{p}') \right)^{n-1} |0\rangle \\
= \;& \alpha(\mathbf{p})\, \frac{1}{(n-1)!} \left(\int \frac{d^3 p'}{(2\pi)^3}\, \alpha(\mathbf{p}')\, a^\dagger(\mathbf{p}') \right)^{n-1} |0\rangle
\end{aligned} \tag{A.3}$$

and similarly

$$\begin{aligned}
& \langle 0|\, \frac{1}{n!} \left(\int \frac{d^3 p'}{(2\pi)^3}\, \alpha^*(\mathbf{p}')\, a(\mathbf{p}') \right)^n a^\dagger(\mathbf{p}) \\
= \;& \langle 0|\, \frac{1}{(n-1)!} \left(\int \frac{d^3 p'}{(2\pi)^3}\, \alpha^*(\mathbf{p}')\, a(\mathbf{p}') \right)^{n-1} \alpha^*(\mathbf{p})\,.
\end{aligned} \tag{A.4}$$

Now we can calculate

$$\begin{aligned}
\langle \alpha|\alpha\rangle \;=\; & \frac{1}{\mathcal{N}^2(\alpha)} \sum_{n=0}^{\infty} \left(\frac{1}{n!} \right)^2 \int \prod_{i=1}^{\infty} \left(\frac{d^3 p_i}{(2\pi)^3} \frac{d^3 p_i'}{(2\pi)^3}\, \alpha^*(\mathbf{p_i})\, \alpha(\mathbf{p_i'}) \right) \cdot \ldots \\
& \ldots \cdot \langle 0| a(\mathbf{p_1}) \cdots a(\mathbf{p_n})\, a^\dagger(\mathbf{p_n'}) \cdots a^\dagger(\mathbf{p_1'}) |0\rangle \\
& .
\end{aligned} \tag{A.5}$$

With

$$
\begin{aligned}
& \langle 0|a(\mathbf{p_1})\cdots a(\mathbf{p_n})\, a^\dagger(\mathbf{p'_n})\cdots a^\dagger(\mathbf{p'_1})|0\rangle \\
= \;& \langle 0|a(\mathbf{p_1})\cdots\Big(\Big[a(\mathbf{p_n}), a^\dagger(\mathbf{p'_n})\Big] + a^\dagger(\mathbf{p'_n})\, a(\mathbf{p_n})\Big)\cdots a^\dagger(\mathbf{p'_1})|0\rangle \\
= \;& (2\pi)^3\,\delta(\mathbf{p_n}-\mathbf{p'_n})\;\langle 0|a(\mathbf{p_1})\cdots a(\mathbf{p_{n-1}})\, a^\dagger(\mathbf{p'_{n-1}})\cdots a^\dagger(\mathbf{p'_1})|0\rangle + ... \\
& ... + \langle 0|a(\mathbf{p_1})\cdots a(\mathbf{p_{n-1}})\, a^\dagger(\mathbf{p'_n})\, a(\mathbf{p_n})\, a^\dagger(\mathbf{p'_{n-1}})\cdots a^\dagger(\mathbf{p'_1})|0\rangle \\
\vdots \quad & \text{(continue normal ordering)} \\
= \;& (2\pi)^3 \sum_{i=1}^{n} \langle 0|a(\mathbf{p_1})\cdots a(\mathbf{p_{n-1}})\, a^\dagger(\mathbf{p'_n})\cdots \widehat{a^\dagger(\mathbf{p'_i})}\cdots a^\dagger(\mathbf{p'_1})|0\rangle\;\cdot\delta(\mathbf{p_n}-\mathbf{p'_i})\,,
\end{aligned}
\tag{A.6}
$$

where

$$
\widehat{a^\dagger(\mathbf{p'_i})} = 1\,. \tag{A.7}
$$

Finally we get

$$
\begin{aligned}
\langle\alpha|\alpha\rangle \;=\;& \frac{1}{\mathcal{N}^2(\alpha)}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{(n-1)!}\int\frac{\mathrm{d}^3 p_n}{(2\pi)^3}\alpha^*(\mathbf{p_n})\,\alpha(\mathbf{p_n})\cdot ... \\
& ... \cdot\int\prod_{i=1}^{n-1}\left(\frac{\mathrm{d}^3 p_i}{(2\pi)^3}\frac{\mathrm{d}^3 p'_i}{(2\pi)^3}\,\alpha^*(\mathbf{p_i})\,\alpha(\mathbf{p'_i})\right)\langle 0|a(\mathbf{p_1})\cdots a(\mathbf{p_{n-1}})\, a^\dagger(\mathbf{p'_{n-1}})\cdots a^\dagger(\mathbf{p'_1})|0\rangle \\
\vdots \quad & \text{(continue normal ordering)} \\
=\;& \frac{1}{\mathcal{N}^2(\alpha)}\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\int\frac{\mathrm{d}^3 p}{(2\pi)^3}\alpha^*(\mathbf{p})\,\alpha(\mathbf{p})\Big)^n \\
=\;& \frac{1}{\mathcal{N}^2(\alpha)}\exp\left(\int\frac{\mathrm{d}^3 p}{(2\pi)^3}|\alpha(\mathbf{p})|^2\right) \\
\Rightarrow \mathcal{N}(\alpha) \;=\;& \exp\left(\frac{1}{2}\int\frac{\mathrm{d}^3 p}{(2\pi)^3}|\alpha(\mathbf{p})|^2\right)\,.
\end{aligned}
\tag{A.8}
$$