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# Quantum Field Theory 1 – Problem set 1

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Suggested reading before solving these problems: Chapters 2.1 and 2.2. in the script and/or Chapters 2.1 to 2.2 of *Peskin & Schroeder*.

## Problem 1: Lagrangian “String Theory”

Consider a series of one-dimensional coupled oscillators  $y_i$ ,  $i = 1, \dots, N$  with distance  $a$ , boundary conditions  $y_0 = y_{N+1} = 0$ , and the Lagrange function

$$L = \sum_{i=1}^N \frac{1}{2} m \dot{y}_i^2 - \sum_{i=0}^N \frac{1}{2} t \left( \frac{y_{i+1} - y_i}{a} \right)^2.$$

Show that the Lagrange function becomes that of a (clamped) string

$$L = \int_0^R dx \left\{ \frac{1}{2} \sigma \left( \frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} \tau \left( \frac{\partial y}{\partial x} \right)^2 \right\}$$

in the limit  $N \rightarrow \infty, a \rightarrow 0$  with  $R = N \cdot a$  fixed. Here  $\sigma = m/a$  is the mass per unit length and  $\tau = t/a$  is the string tension. By expanding the displacement as a Fourier expansion in the form

$$y(x, t) = \sqrt{\frac{2}{R}} \sum_{n=1}^{\infty} q_n(t) \sin \left( \frac{n\pi x}{R} \right)$$

show that

$$L = \sum_{n=1}^{\infty} \left\{ \frac{1}{2} \sigma \dot{q}_n^2 - \frac{1}{2} \tau \left( \frac{n\pi}{R} \right)^2 q_n^2 \right\}.$$

Use the variational principle with this form of the Lagrangian to obtain the Euler-Lagrange equations

$$\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{q}_n} \right) - \frac{\partial L}{\partial q_n} = 0.$$

Hence show that the string is equivalent to an infinite set of harmonic oscillators with frequencies

$$\omega_n = \sqrt{\frac{\tau}{\sigma}} \frac{n\pi}{R}.$$

What happens in the limit  $R \rightarrow \infty$ ?

## Problem 2: Complex scalar field

Consider the following action for a complex scalar field

$$S = \int d^4x \mathcal{L} = \int d^4x \left\{ \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2 \right\}.$$

It is easiest to consider  $\phi$  and  $\phi^*$  as independent, rather than the real and imaginary parts of  $\phi$ .

- a) Derive the Euler-Lagrange equations for  $\phi$  and  $\phi^*$ .
- b) Show that  $S$  is invariant under the infinitesimal transformation

$$\begin{aligned} \phi(x) &\rightarrow (1 + i\alpha) \phi(x) \\ \phi^*(x) &\rightarrow (1 - i\alpha) \phi^*(x). \end{aligned} \tag{1}$$

- c) Derive an expression for the Noether current  $j^\mu = (j^0, \mathbf{j})$  associated with this symmetry transformation and show that it is conserved for fields  $\phi, \phi^*$  that satisfy the Euler-Lagrange equations.
- d) Show that the invariance of  $S$  under infinitesimal space and time translations leads to four conserved currents. Give interpretations for the components of the energy-momentum tensor

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \partial_\nu \phi^* - \mathcal{L} \delta^\mu{}_\nu$$

and derive explicit expressions.