Quantum Field Theory 1 – Problem set 7

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Suggested reading before solving these problems: Chapters 4.1 in the script and/or Chapters 3.1 of *Peskin & Schroeder*.

Problem 1: Infinitesimal and finite Lorentz transformations

Under a Lorentz transformation the coordinates transform as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu ,$$

with a matrix Λ that leaves the metric tensor unchanged

$$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma} .$$

a) An infinitesimal Lorentz transformation has the form

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu \, .$$

Show that $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

b) Write down the matrix $\omega^\mu{}_\nu$ that correspond to a rotation through an infinitesimal angle ϑ around the z-axis. Do the same for a boost along the z-axis by an infinitesimal velocity v. In both cases, check the validity of the relation shown in part a).

c) By exponentiating, deduce the form of $\Lambda^\mu{}_\nu$ for a finite rotation around or a finite boost along the z-axis.

Problem 2: Generators of the Lorentz group

a) Verify that the infinitesimal Lorentz transformations found in problem 1 can be written as

$$\Lambda = \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta} M^{\alpha\beta}$$

where the matrix $M^{\alpha\beta}$ has the components

$$(M^{\alpha\beta})^\mu{}_\nu = i(\eta^{\mu\alpha}\delta^\beta_\nu - \eta^{\mu\beta}\delta^\alpha_\nu).$$

b) Show that the generators $M^{\alpha\beta}$ of infinitesimal Lorentz transformations satisfy the commutation relations (Lie algebra brackets)

$$[M^{\alpha\beta}, M^{\gamma\delta}] = i\left(\eta^{\beta\gamma}M^{\alpha\delta} + \eta^{\alpha\delta}M^{\beta\gamma} - \eta^{\alpha\gamma}M^{\beta\delta} - \eta^{\beta\delta}M^{\alpha\gamma}\right).$$

c) Show that

$$[M^{01}, M^{23}] = 0$$

and that this holds for all permutations of $0, 1, 2, 3$.

d) The generators of rotations are

$$J_1 = M^{23}, \quad J_2 = M^{31}, J_3 = M^{12}.$$

Check that the J_k satisfy the angular momentum commutation relation

$$[J_i, J_j] = i\,\epsilon_{ijk}\,J_k,$$

e) The generators of Lorentz boosts are

$$K_1 = M^{10}, \quad K_2 = M^{20}, \quad K_3 = M^{30}.$$

Check the commutation laws

$$[J_i, K_j] = i\,\epsilon_{ijk}\,K_k, \quad [K_i, K_j] = -\,i\,\epsilon_{ijk}\,J_k.$$

f) Show that the differential operators

$$\tilde{M}^{\alpha\beta} = i\left(x^\alpha\partial^\beta - x^\beta\partial^\alpha\right)$$

(with $\partial_\mu = \frac{\partial}{\partial x^\mu}$) satisfy the commutation relations of the generators of Lorentz transformations in part b).

Problem 3: Transformation of fields

A scalar field $\phi(x)$ transforms under a Lorentz transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

according to

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x).$$

In contrast, a vector field $A^\mu(x)$ transforms as

$$A^\mu(x) \to A'^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x).$$

a) What is the transformation behaviour of a tensor field $T^{\mu\nu}(x)$ or more general $T^{\alpha_1\alpha_2\ldots\alpha_n}{}_{\beta_1\beta_2\ldots\beta_m}(x)$?

b) An example for a tensor field is the electromagnetic field strength $F^{\mu\nu}$ while the current j^μ is a vector field. Show that Maxwells equation

$$\partial_\mu F^{\mu\nu} = j^\nu$$

is invariant under Lorentz transformations.

c) Show that the following special "field configurations" are invariant:

$$\phi(x) = \text{const.}, \quad A^\mu(x) = x^\mu, \quad T^{\mu\nu}(x) = \eta^{\mu\nu}, \quad T^\mu{}_\nu(x) = x^\mu x_\nu.$$