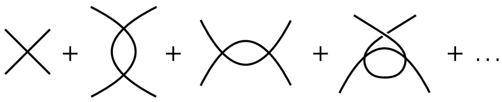
Quantum Field Theory 1 – Problem set 10

Lectures: Jan Pawlowski pawlowski@thphys.uni-heidelberg.de
Tutorials: Michael Scherer scherer@thphys.uni-heidelberg.de
Institut für Theoretische Physik, Uni Heidelberg due date: 11 January 2017

It ain't what you don't know that gets you into trouble. It's what you know for sure that just ain't so. - Mark Twain

Problem 1: High-energy cutoff and scattering amplitudes

In the lecture and on problem set 9, we have encountered divergent integrals appearing in our calculations. In this exercise, you can develop a first idea of how to deal with these infinities. We will not worry too much about every detail of the calculation. Instead, we try to emphasise the concepts. Recall that in the process of calculating the scattering amplitude $\mathcal{M}$ for 2-2 scattering in scalar quantum field theory, you find the diagrams, cf. script Sec. 3.4, Eq. (3.95):



a) Use the Feynman rules, cf. script Sec. 3.2, p. 67f, to show that the second diagram gives the following contribution $\mathcal{M}_s$ to the scattering amplitude $\mathcal{M}$

$$\mathcal{M}_s = \frac{1}{2}(-i\lambda)^2 i^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} \frac{1}{(p_1 + p_2 - p)^2 - m^2 + i\epsilon} \,. \tag{1}$$

b) Also, write down the other two loop-integrals, explicitely.

c) Argue that the integral exhibits a logarithmic divergence associated with large values of p, i.e. an *ultraviolet divergence.*

Time to pause and reflect: Integrating out all momenta up to infinity implicitly makes the assumption that the theory is valid up to arbitrarily high energy scales. This is presumptuous. It is more reasonable to assume, that the theory is only valid up to some threshold energy[1] which we will call Λ. Therefore, the integral in Eq. (1) should only be integrated up to Λ which defines the physical limit of validity of the theory, i.e. the momentum integration is *cut off.* The result of the integral in Eq. (1) with cutoff Λ is $2iC\log(\Lambda^2/s)$ with $s = (p_1 + p_2)^2$, some constant C and we assume that $m^2 \ll s$. Here, we will not bother about the calculation of C.

d) Gather all the contributions to order $\mathcal{O}(\lambda^2)$ to the scattering amplitude to show

$$\mathcal{M} = -i\lambda + iC\lambda^2 L + \mathcal{O}(\lambda^3) \quad \text{with} \quad L = \log(\Lambda^2/s) + \log(\Lambda^2/t) + \log(\Lambda^2/u)\,, \tag{2}$$

where $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$. Note, that $\mathcal{M}$ is not divergent anymore but instead depends logarithmically on the cutoff Λ.

[1] Recall the discussion of the Casimir effect where for high frequencies the plates become transparent.

Now, suppose an experimentalist measures the coupling for scalar-scalar interaction by scattering a scalar particle off another at a particular energy with a particular scattering angle as given by the kinematic parameters $s_{\rm exp}, t_{\rm exp}, u_{\rm exp}$ and finds the value $\lambda_{\rm exp}$. Let's think about how $\lambda_{\rm exp}$ is related to our theoretical parameter λ: The experimentalist does not measure individual Feynman diagrams! In fact, the measured $\lambda_{\rm exp}$ already contains all diagrams, so

$$-i\lambda_{\rm exp} = -i\lambda + iC\lambda^2 L_{\rm exp} + \mathcal{O}(\lambda^3)\,. \tag{3}$$

e) Show that Eq. (3) can be solved for λ to give

$$-i\lambda = -i\lambda_{\rm exp} - iC\lambda_{\rm exp}^2 L_{\rm exp} + \mathcal{O}(\lambda_{\rm exp}^3)\,, \tag{4}$$

f) Plug Eq. (4) into Eq. (2) and show that the scattering amplitude is

$$\mathcal{M} = -i\lambda_{\rm exp} + iC\lambda_{\rm exp}^2 \left[\log(s_{\rm exp}/s) + \log(t_{\rm exp}/t) + \log(u_{\rm exp}/u)\right] + \mathcal{O}(\lambda_{\rm exp}^3)\,.$$

Check that the manipulations are allowed to the order of approximation indicated.

Note, that in the expression for the scattering amplitude we have now traded the purely theoretical parameter λ for a measurable quantity – the physical coupling constant $\lambda_{\rm exp}$ – and as a result the cutoff Λ has disappeared from the scattering amplitude.

Problem 2: Cutoff and coupling constant

We add another perspective: The cutoff independence of $\mathcal{M}$ requires the parameter λ to be a function of Λ, i.e. if somebody changes Λ you have to adjust λ to get a cutoff-independent $\mathcal{M}$. Let's find out how this adjustment works:

a) Change Λ to $e^\epsilon\Lambda$ in Eq. (2) and calculate the induced change of the scattering amplitude $\delta\mathcal{M}$.

b) To have a cutoff-independent $\mathcal{M}$, we demand $\delta\mathcal{M} = 0$ to the order indicated in λ. Show that this condition implies $\delta\lambda = 6\epsilon\, C\lambda^2 + \mathcal{O}(\lambda^3)$. Show that this can be expressed in terms of a differential equation

$$\Lambda\frac{d\lambda}{d\Lambda} = 6C\lambda^2 \tag{5}$$

where higher-order terms have been neglected.

Advanced things you could think about over the holidays

- How to calculate the integral in Eq. (1) with cutoff Λ and the constant C? – see for example 'Quantum Field Theory in a Nutshell' by A. Zee, cf. Part III.1.
- Take Eq. (5) and suppose you have managed to fix $\lambda = \lambda_0 > 0$ at some cutoff $\Lambda = \Lambda_0$. What happens, if you try to send the cutoff to higher and higher energies?
- The quote from the top of the first page.