Quantum Field Theory 1 – Tutorial 2

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Problem 1: Fourier transform and Dirac distribution

Consider a three-dimensional cubus $x_i \in (-L/2, L/2), i = 1, 2, 3$ with volume $V = L^3$ and periodic boundary conditions. A (complex) field configuration $\phi(\boldsymbol{x})$ can be written as

$$\phi(\boldsymbol{x}) = \frac{1}{V} \sum_{l,m,n} e^{i\boldsymbol{p}_{lmn}\boldsymbol{x}}\, \tilde{\phi}_{lmn} \tag{1}$$

with $\boldsymbol{p}_{lmn} = \left(\frac{2\pi l}{L}, \frac{2\pi m}{L}, \frac{2\pi n}{L}\right)$.

a) What is the range of the indices l, m, n?

b) Show that

$$\int_V d^3x\, e^{i(\boldsymbol{p}_{lmn} - \boldsymbol{p}_{l'm'n'})\boldsymbol{x}} = V \delta_{ll'}\delta_{mm'}\delta_{nn'} \tag{2}$$

and hence

$$\tilde{\phi}_{lmn} = \int_V d^3x\, e^{-i\boldsymbol{p}_{lmn}\boldsymbol{x}}\, \phi(\boldsymbol{x}).$$

c) Consider now the limit $L \to \infty$. Show that Eq. (1) becomes

$$\phi(\boldsymbol{x}) = \frac{1}{(2\pi)^3} \int d^3p\, e^{i\boldsymbol{p}\boldsymbol{x}}\, \tilde{\phi}(\boldsymbol{p}).$$

d) By comparing Eq. (2) to its infinite volume limit

$$\int d^3x\, e^{i(\boldsymbol{p}-\boldsymbol{p}')\boldsymbol{x}} = (2\pi)^3\, \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$$

convince yourself that

$$(2\pi)^3\, \delta^{(3)}(\boldsymbol{q})\big|_{\boldsymbol{q}=0} = \lim_{L\to\infty} V.$$