Quantum Field Theory 1 – Tutorial 3

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Problem 1: Lorentz invariant quantities

In the lecture we have used the following important identity for the three-momentum integral

$$\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\boldsymbol{p}}} f(p) \;=\; \int \frac{d^4p}{(2\pi)^4}\, 2\pi\,\delta(p^2-m^2)\,\Theta(p_0) f(p)\,. \tag{1}$$

Use the delta function identity

$$\delta(g(x)-g(a)) \;=\; \frac{1}{|g'(a)|}\,\delta(x-a)\,.$$

to derive Eq. (1). Further, convince yourself that this implies that the three-momentum integral is invariant under Lorentz transformations Λ if $f(\Lambda p) = f(p)$.

Remark: We consider proper, orthochronous Lorentz transformations $\Lambda \in SO(1,3)$.

Optional: In the lecture we have also introduced conveniently normalized one-particle states with momentum $\boldsymbol{p}$ as $|\boldsymbol{p}\rangle = \sqrt{2\omega_{\boldsymbol{p}}} a^\dagger_{\boldsymbol{p}}|0\rangle$ with $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2+m^2}$ (see Eq. (2.65) in the script). Consider a Lorentz boost in the p_3 direction:

$$p_3' = \gamma(p_3 + \beta\, E)\,, \quad \text{and} \quad E' = \gamma(E + \beta\, p_3)\,.$$

Use the delta function identity from above to show that the boost gives

$$\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) = \delta^{(3)}(\boldsymbol{p}'-\boldsymbol{q}')\frac{E'}{E}\,.$$

This non-trivial transformation property is related to the fact that volumes are not invariant under boosts due to Lorentz contraction ($V \to V/\gamma$). Instead, the quantity $E\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$ is invariant under boosts. More generally, $E\,\delta^{(3)}\delta(\boldsymbol{p}-\boldsymbol{q})$ is Lorentz invariant which also makes $\langle \boldsymbol{p}|\boldsymbol{q}\rangle = 2\omega_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$ a Lorentz invariant quantity.