Quantum Field Theory 1 – Tutorial 13

Lectures: Jan Pawlowski pawlowski@thphys.uni-heidelberg.de
Tutorials: Michael Scherer scherer@thphys.uni-heidelberg.de
Institut für Theoretische Physik, Uni Heidelberg tutorial date: 25 January 2017

Problem 1: Asymptotic expansion

Let's consider a caricature of the integrals that we have to calculate when setting up a perturbation expansion of an interacting QFT:

$$I(\lambda) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 - \lambda x^4} \,. \tag{1}$$

For small $\lambda \ll 1$ and $\lambda > 0$ we would consider it 'natural' to treat the interaction λx^4 perturbatively.

a) Develop the expansion $I(\lambda) = \sum_{n=0}^{\infty} \lambda^n I_n$. Explicitly, write down the $\lambda^n I_n$ and carry out the integral[1].

b) Use Stirling's approximation, $n! \sim n^n e^{-n}$ for $n \gg 1$, to show that $\lambda^n I_n \sim \left(-\frac{16\lambda n}{e}\right)^n$ for large n.

c) When does the perturbative expansion start to diverge?

This simple example tells us that the series expansion in the 'small parameter' λ does not exist. We note that this does not imply that the idea of the perturbative expansion should be abandoned. In fact, partial resummations of the perturbative series up to an $n_{\max}$ can yield excellent approximations to the exact result for $I(\lambda)$.

Optional:

d) Estimate the error[2] of a partial resummation up to order $n_{\max}$,

$$\left| I(g) - \sum_{n=0}^{n_{\max}} \lambda^n I_n \right| \,.$$

e) For a given λ, estimate the value of $n_{\max}$ where the error is minimal.

[1] Use $\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} x^2 n = \sqrt{\frac{\pi}{\alpha}} \frac{(2n-1)!!}{(2\alpha)^n}$ and $(2n-1)!! = (2n)!/(2^n n!)$.

[2] Use $\left|e^{-\lambda x^4} - \sum_{n=0}^{n_{\max}} \frac{(-\lambda x^4)^n}{n!}\right| \le \frac{(\lambda x^4)^{n_{\max}+1}}{(n_{\max}+1)!}$.