Quantum Field Theory 1 – Problem set 2

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Suggested reading before solving these problems: Chapters 2.3 in the script and/or Chapters 2.3 to 2.4 of *Peskin & Schroeder*.

Problem 1: Commutation relations

For a real scalar field $\phi(x)$ the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

The canonical momentum density is $\pi = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}$. The theory is quantized by promoting ϕ and π to operators in the Schrödinger picture with the commutation relations

$$\begin{aligned}
[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] &= i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \\
[\phi(\boldsymbol{x}), \phi(\boldsymbol{y})] &= [\pi(\boldsymbol{x}), \pi(\boldsymbol{y})] = 0.
\end{aligned}$$

Introduce now the operators a and $a^\dagger$ by

$$\begin{aligned}
\phi(\boldsymbol{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left\{a(\boldsymbol{p})\,e^{i\boldsymbol{px}} + a^\dagger(\boldsymbol{p})\,e^{-i\boldsymbol{px}}\right\}, \\
\pi(\boldsymbol{x}) &= -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left\{a(\boldsymbol{p})\,e^{i\boldsymbol{px}} - a^\dagger(\boldsymbol{p})\,e^{-i\boldsymbol{px}}\right\}
\end{aligned}$$

and derive the commutation relations

$$[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})], \quad [a(\boldsymbol{p}), a(\boldsymbol{q})], \quad [a^\dagger(\boldsymbol{p}), a^\dagger(\boldsymbol{q})].$$

Problem 2: Complex scalar field: quantization

Consider the Lagrangian density for a free complex scalar field

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi.$$

a) Show that the canonical momenta of ϕ and ϕ^* are

$$\pi = \dot{\phi}^*, \quad \pi^* = \dot{\phi}.$$

and derive an expression for the Hamiltonian H.

b) Proceed to quantization by promoting ϕ, ϕ^* and π, π^* to operators $\phi, \phi^\dagger$ and $\pi, \pi^\dagger$ (in the Schrödinger picture). What would you postulate as their commutation relations?

c) Introduce now creation and annihilation operators by writing

$$\begin{aligned} \phi(\boldsymbol{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left\{ a(\boldsymbol{p})\, e^{i\boldsymbol{p}\boldsymbol{x}} + b^\dagger(\boldsymbol{p})\, e^{-i\boldsymbol{p}\boldsymbol{x}} \right\}, \\ \pi(\boldsymbol{x}) &= -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left\{ -a^\dagger(\boldsymbol{p})\, e^{-i\boldsymbol{p}\boldsymbol{x}} + b(\boldsymbol{p})\, e^{i\boldsymbol{p}\boldsymbol{x}} \right\}. \end{aligned}$$

Why do we now need operators $b, b^\dagger$ in addition to $a, a^\dagger$? Convince yourself that the commutation relations

$$\begin{aligned} [a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})] &= [b(\boldsymbol{p}), b^\dagger(\boldsymbol{q})] = (2\pi)^3\, \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \\ [a(\boldsymbol{p}), a(\boldsymbol{q})] &= [b(\boldsymbol{p}), b(\boldsymbol{q})] = 0, \\ [a(\boldsymbol{p}), b(\boldsymbol{q})] &= [a(\boldsymbol{p}), b^\dagger(\boldsymbol{q})] = 0 \end{aligned}$$

are consistent with the ones postulated in part b).

d) Show that the Hamiltonian can be written as

$$H = \int \frac{d^3p}{(2\pi)^3}\, \omega_{\boldsymbol{p}} \left\{ a^\dagger(\boldsymbol{p}) a(\boldsymbol{p}) + b^\dagger(\boldsymbol{p}) b(\boldsymbol{p}) \right\} + \text{const.}$$

Why is the positive sign in front of the $b^\dagger b$ term important? What is the physical interpretation of b and $b^\dagger$?

e) Switch now to the Heisenberg picture

$$\phi_H(t, \boldsymbol{x}) = e^{iHt} \phi(\boldsymbol{x}) e^{-iHt}.$$

Show that

$$\begin{aligned} e^{iHt}\, a(\boldsymbol{p})\, e^{-iHt} &= a(\boldsymbol{p})\, e^{-i\omega_{\boldsymbol{p}} t}, & e^{iHt}\, a^\dagger(\boldsymbol{p})\, e^{-iHt} &= a^\dagger(\boldsymbol{p})\, e^{i\omega_{\boldsymbol{p}} t}, \\ e^{iHt}\, b(\boldsymbol{p})\, e^{-iHt} &= b(\boldsymbol{p})\, e^{-i\omega_{\boldsymbol{p}} t}, & e^{iHt}\, b^\dagger(\boldsymbol{p})\, e^{-iHt} &= b^\dagger(\boldsymbol{p})\, e^{i\omega_{\boldsymbol{p}} t}, \end{aligned} \tag{1}$$

and therefore

$$\phi_H(t, \boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left\{ a(\boldsymbol{p})\, e^{-ipx} + b^\dagger(\boldsymbol{p})\, e^{ipx} \right\}.$$