Quantum Field Theory 1 – Problem set 3

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Suggested reading before solving these problems: Chapter 2.3, 3.1 in the script and/or Chapters 2.2, 2.3 and 2.4 of *Peskin & Schroeder*.

Problem 1: A commutation relation for the three-momentum operator

Consider a real scalar field with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

and conjugate momentum $\pi = \partial\mathcal{L}/\partial\dot{\phi}$. You have learned how to quantize the system by promoting ϕ and π to operators with the commutation relations

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \tag{1}$$

$$[\phi(\boldsymbol{x}), \phi(\boldsymbol{y})] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{y})] = 0. \tag{2}$$

The associated Hamiltonian and the three-momentum operators read

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{m^2}{2}\phi^2\right]$$

$$\boldsymbol{P} = \int d^3x \left(\pi \, \boldsymbol{\nabla}\phi\right)$$

Show that these operators commute: $[H, \boldsymbol{P}] = 0$ by *only* using the commutation relations (1) and (2).

Problem 2: Charge of a complex scalar field

Consider the Lagrangian density for a free complex scalar field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi.$$

and define the associated conjugate momenta π and π^*. The Noether theorem leads to a conserved charge, given in terms of 0-component of the Noether current:

$$Q \equiv \int d^3x \, j^0 \, .$$

For a complex scalar field, the four-vector associated with the current j reads

$$j^\mu \;=\; i\,\left[(\partial^\mu\phi)^*\,\phi - \phi^*\,(\partial^\mu\phi)\right],$$

from which an expression for the corresponding charge Q can be easily obtained.

The theory is quantised by promoting ϕ, ϕ^* and their conjugate momenta to operators. To that end it is convenient to introduce the creation and annihilation operators

$$\phi(\boldsymbol{x}) \;=\; \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left\{a(\boldsymbol{p})\,e^{i\boldsymbol{p}\boldsymbol{x}} + b^\dagger(\boldsymbol{p})\,e^{-i\boldsymbol{p}\boldsymbol{x}}\right\}, \tag{3}$$

$$\pi(\boldsymbol{x}) \;=\; -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left\{b(\boldsymbol{p})\,e^{i\boldsymbol{p}\boldsymbol{x}} - a^\dagger(\boldsymbol{p})\,e^{-i\boldsymbol{p}\boldsymbol{x}}\right\}. \tag{4}$$

with commutation relations

$$\begin{aligned}
[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})] &= [b(\boldsymbol{p}), b^\dagger(\boldsymbol{q})] = (2\pi)^3\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}),\\
[a(\boldsymbol{p}), a(\boldsymbol{q})] &= [b(\boldsymbol{p}), b(\boldsymbol{q})] = 0,\\
[a(\boldsymbol{p}), b(\boldsymbol{q})] &= [a(\boldsymbol{p}), b^\dagger(\boldsymbol{q})] = 0\;.
\end{aligned}$$

Express the charge Q in terms of the operators a, $a^\dagger$ and b, $b^\dagger$, carrying out all the details of the calculation.