Quantum Field Theory 1 – Problem set 8

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Suggested reading before solving these problems: Chapters 4.2 in the script and/or Chapters 3.2 of *Peskin & Schroeder*.

Problem 1: Lorentz symmetry $SO(3,1)$ and $SL(2,\mathbb{C})$

Consider te 2×2 matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For a space-time coordinate x^μ consider the matrix

$$\hat{x} = x_\mu \sigma^\mu = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix}. \tag{1}$$

a) Show that every hermitian 2×2 matrix can be written in the form (1) for some real x^μ.

b) Show that $\det \hat{x} = x_\nu x^\nu$ and that this implies

$$\eta_{\mu\nu} x^\mu y^\nu = \frac{1}{4} \left[\det(\hat{x} + \hat{y}) - \det(\hat{x} - \hat{y}) \right].$$

c) Show for a complex 2×2 matrix N with unit determinant, $N \in SL(2, \mathbb{C})$, that

$$\hat{x}' = N \hat{x} N^\dagger$$

is again of the form (1) with x' linear in x,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu.$$

d) By considering the relations in b) for the matrices $\hat{x}' = N \hat{x} N^\dagger$ and $\hat{y}' = N \hat{y} N^\dagger$ show that

$$\eta_{\mu\nu} x'^\mu y'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\tau x^\rho x^\tau = \eta_{\mu\nu} x^\mu y^\nu$$

and that Λ has therefore the properties of a Lorentz-transformation.

e) Define $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$ and show the identities

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu} \mathbb{1}_2,$$

$$\mathrm{tr}(\sigma^\mu \bar{\sigma}^\nu) = 2\eta^{\mu\nu}.$$

f) For a given $N \in SL(2, \mathbb{C})$ show that the corresponding matrix Λ is given by

$$\Lambda^\mu{}_\nu = \frac{1}{2} \mathrm{Tr} \left(\bar{\sigma}^\mu N \sigma_\nu N^\dagger \right). \tag{2}$$

From Eq. (2) you can see that there is a unique matrix $\Lambda \in SO(3,1)$ for every matrix $N \in SL(2, \mathbb{C})$. On the other side, N and $-N$ correspond to the same Λ. In fact, there is a isomorphism from $SL(2, \mathbb{C})/\mathbb{Z}_2$ (the complex 2×2 matrices with N and $-N$ identified) to the orthochronous Lorentz group $SO(3,1)^\uparrow$ consisting of of matrices Λ that conserve the metric tensor with $\Lambda^0{}_0 > 0$ and $\det \Lambda = 1$.

Problem 2: Dirac algebra

a) Show that the Dirac gamma matrices

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$$

satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}_4.$$

b) A representation of the Lorentz algebra (see Sheet 7) is given by

$$S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu].$$

Calculate the explicit form of S^{0i} and S^{ij}.

c) With the notation of Sheet 7, problem 2, show that

$$[\gamma^\mu, S^{\rho\sigma}] = (M^{\rho\sigma})^\mu{}_\nu \gamma^\nu$$

and that this implies the infinitesimal form of

$$\Lambda_{\frac{1}{2}}^{-1} \, \gamma^\mu \, \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \gamma^\nu$$

where $\Lambda_{\frac{1}{2}} = \exp(-\frac{i}{2} \omega_{\mu\nu} S^{\mu\nu})$ is a 4×4 matrix in spinor space and $\Lambda = \exp(-\frac{i}{2} \omega_{\alpha\beta} M^{\alpha\beta})$ is a 4×4 matrix in Lorentz space.